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XXVII. *Table of Zonal Spherical Harmonics, calculated by Messrs. C. E. Holland, P. R. Jones, and C. G. Lamb With a short Explanation and some Illustrations of its use by Professor JOHN PERRY, D.Sc., F.R.S.**

[Plate VI.]

I HAD intended merely to present to the Society for publication a table of Zonal Spherical Harmonics. But some Members whom I have consulted seem to think that I ought also to give one or two examples of the practical use of such a table. The Members of the Society will, I hope, pardon my putting before them one or two elementary examples.

The use of Spherical Harmonics in the numerical solution of practical problems is almost unknown, I believe, except at the Finsbury Technical College, where, every year, I have been accustomed to make some Electrical Engineering students work a few common examples. My students have for some years been in possession of tables of zonal harmonics, but this year I have thought it well to make the tables more complete and to get them published for the general use of students of practical physics.

I have been told that many of the users of such a table would be glad of a few statements of the general principles underlying its use. For the proofs of these statements readers

* Read November 14, 1890.

are referred to Mathematical treatises. Many readers will be satisfied with the treatment of the subject in Mr. Ferrer's excellent treatise, which is, however, written only for beginners.

In problems on Heat Conduction (V being temperature), on Hydrodynamics of incompressible fluids (V being velocity-potential*), in Electrostatics (V being electric potential), in Magnetism (V being magnetic potential), and in many other applications of Physics, we require to find V a function of x, y, z which shall satisfy the equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0, \quad \dots \quad (1)$$

or, as it is usually written, $\nabla^2 V = 0 \dots (1)$, and which shall also satisfy certain other conditions. Now there are many kinds of function which satisfy equation (1). The definition of a Spherical Harmonic is "a *homogeneous* function of x, y , and z , which satisfies equation (1)."

If such a function can be found, say of the i th degree, and if we divide it by r^{2i+1} where $r^2 = x^2 + y^2 + z^2$, it can be proved that the resulting expression will also satisfy (1), where i may be a positive or negative integer or fraction.

Now if a Spherical Harmonic of degree i (generally called a Solid Spherical Harmonic) be divided by r , we get what is called a *Surface* Spherical Harmonic of degree i .

It was shown by Green that if there is a function V which satisfies equation (1) at every point of any given surface, then it is the only function which satisfies (1) throughout space; and there is always a function V obtainable which satisfies (1). It is the characteristic property of a surface spherical harmonic distribution of density of attracting matter on a spherical surface, that it produces a similar and similarly placed spherical harmonic distribution of potential

* When there exists a velocity-potential V in a portion of fluid, we mean that the velocity of the fluid at any place resolved in the direction s is

$$-\frac{dV}{ds}.$$

When the motion is "rotational," as in the wheel of a centrifugal pump or turbine, a velocity-potential does not exist. In any portion of a frictionless fluid, if there is irrotationality, that is, if there is a velocity-potential, the property cannot be destroyed.

over any concentric spherical surface throughout space, external and internal.

Instead of using x , y , and z coordinates we may of course use r , θ , and ϕ coordinates.

In a great number of practical cases V is symmetrical about an axis, and a symmetrical spherical harmonic is said to be a Zonal Spherical Harmonic. Taking the axis of symmetry as the axis of z , V is a function of z and $\sqrt{x^2 + y^2}$. Or, in polar coordinates, V is a function of r and θ .

Let O be a point in the axis, the origin of coordinates; let the distance of any point P from O be called r , let the angle between OP and the axis be called θ , then in any distribution which has an axis of symmetry we need only to know r and θ . And over any spherical surface whose centre is O , the distribution will be a function of θ . Any zonal surface spherical harmonic is then merely a function of θ , and I give a table of values of these for values of θ differing by 1° from 0° to 90° , up to the harmonic of the seventh degree. These are indicated by P_0 , P_1 , P_2 , &c. P_7 .

The surface harmonic of no degree is 1, and is indicated by P_0 .

The student is referred to Mathematical treatises for the proof that, if μ be written to represent $\cos \theta$, then

$$P_0 = 1,$$

$$P_1 = \mu,$$

$$P_2 = \frac{3\mu^2 - 1}{2},$$

$$P_3 = \frac{5\mu^3 - 3\mu}{2},$$

$$P_4 = \frac{35\mu^4 - 30\mu^2 + 3}{8},$$

$$P_5 = \frac{63\mu^5 - 70\mu^3 + 15\mu}{8},$$

$$P_6 = \frac{231\mu^6 - 315\mu^4 + 105\mu^2 - 5}{16},$$

$$P_7 = \frac{429\mu^7 - 693\mu^5 + 315\mu^3 - 35\mu}{16}.$$

Any function of θ may be expanded in terms of P_0, P_1, P_2 , &c., that is any symmetrical function V may be expanded in a series of Zonal Spherical Harmonics. Take, for example, the powers of $\cos \theta$, it may be shown that

$$\mu^0 = 1 = P_0,$$

$$\mu = P_1,$$

$$\mu^2 = \frac{2}{3} P_2 + \frac{1}{3} P_0,$$

$$\mu^3 = \frac{2}{5} P_3 + \frac{3}{5} P_1,$$

$$\mu^4 = \frac{8}{35} P_4 + \frac{4}{7} P_2 + \frac{1}{5} P_0,$$

$$\mu^5 = \frac{8}{63} P_5 + \frac{4}{9} P_3 + \frac{3}{7} P_1.$$

And also

$$\cos \theta = P_1,$$

$$\cos 2\theta = \frac{4}{3} P_2 - \frac{1}{3} P_0,$$

$$\cos 3\theta = \frac{8}{5} P_3 - \frac{3}{5} P_1,$$

$$\cos 4\theta = \frac{64}{35} P_4 - \frac{16}{21} P_2 - \frac{1}{15} P_0,$$

$$\cos 5\theta = \frac{128}{63} P_5 - \frac{8}{9} P_3 - \frac{1}{7} P_1.$$

There is an easy rule for expanding any function of θ in terms of P_0, P_1, P_2 , &c.

The following table up to P_5 was primarily calculated separately by Messrs. C. E. Holland and P. R. Jones, who checked each other's result by comparison.

Mr. C. G. Lamb applied certain checks to the results of Messrs. Holland and Jones, and then calculated P_6 and P_7 .

It is quite easy to extend the table to P_8 , for it can be proved that there is a law connecting three consecutive harmonics, say the $(n-2)$ th, $(n-1)$ th, and the n th.

$$nP_n = (2n-1)\mu P_{n-1} - (n-1)P_{n-2}.$$

θ .	P_1 .	P_2 .	P_3 .	P_4 .	P_5 .	P_6 .	P_7 .
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	.9998	.9995	.9991	.9985	.9977	.9967	.9955
2	.9994	.9982	.9963	.9939	.9909	.9872	.9829
3	.9986	.9959	.9918	.9863	.9795	.9713	.9617
4	.9976	.9927	.9854	.9758	.9638	.9495	.9329
5	.9962	.9886	.9773	.9623	.9437	.9216	.8961
6	.9945	.9836	.9674	.9459	.9194	.8881	.8522
7	.9925	.9777	.9557	.9267	.8911	.8476	.7986
8	.9903	.9709	.9423	.9048	.8589	.8053	.7448
9	.9877	.9633	.9273	.8803	.8232	.7571	.6831
10	.9848	.9548	.9106	.8532	.7840	.7045	.6164
11	.9816	.9454	.8923	.8238	.7417	.6483	.5461
12	.9781	.9352	.8724	.7920	.6966	.5892	.4732
13	.9744	.9241	.8511	.7582	.6489	.5273	.3940
14	.9703	.9122	.8283	.7224	.5990	.4635	.3219
15	.9659	.8995	.8042	.6847	.5471	.3982	.2454
16	.9613	.8860	.7787	.6454	.4937	.3322	.1699
17	.9563	.8718	.7519	.6046	.4391	.2660	.0961
18	.9511	.8568	.7240	.5624	.3836	.2002	.0289
19	.9455	.8410	.6950	.5192	.3276	.1347	-.0443
20	.9397	.8245	.6649	.4750	.2715	.0719	-.1072
21	.9336	.8074	.6338	.4300	.2156	.0107	-.1662
22	.9272	.7895	.6019	.3845	.1602	-.0481	-.2201
23	.9205	.7710	.5692	.3386	.1057	-.1038	-.2681
24	.9135	.7518	.5357	.2926	.0525	-.1559	-.3095
25	.9063	.7321	.5016	.2465	.0009	-.2053	-.3463
26	.8988	.7117	.4670	.2007	-.0489	-.2478	-.3717
27	.8910	.6908	.4319	.1553	-.0964	-.2869	-.3921
28	.8829	.6694	.3964	.1105	-.1415	-.3211	-.4052
29	.8746	.6474	.3607	.0665	-.1839	-.3503	-.4114
30	.8660	.6250	.3248	.0234	-.2233	-.3740	-.4101
31	.8572	.6021	.2887	-.0185	-.2595	-.3924	-.4022
32	.8480	.5788	.2527	-.0591	-.2923	-.4052	-.3876
33	.8387	.5551	.2167	-.0982	-.3216	-.4126	-.3670
34	.8290	.5310	.1809	-.1357	-.3473	-.4148	-.3409
35	.8192	.5065	.1454	-.1714	-.3691	-.4115	-.3096
36	.8090	.4818	.1102	-.2052	-.3871	-.4031	-.2738
37	.7986	.4567	.0755	-.2370	-.4011	-.3898	-.2343
38	.7880	.4314	.0413	-.2666	-.4112	-.3719	-.1918
39	.7771	.4059	.0077	-.2940	-.4174	-.3497	-.1469
40	.7660	.3802	-.0252	-.3190	-.4197	-.3234	-.1003
41	.7547	.3544	-.0574	-.3416	-.4181	-.2938	-.0534
42	.7431	.3284	-.0887	-.3616	-.4128	-.2611	-.0065
43	.7314	.3023	-.1191	-.3791	-.4038	-.2255	.0398
44	.7193	.2762	-.1485	-.3940	-.3914	-.1878	.0846
45	.7071	.2500	-.1768	-.4062	-.3757	-.1485	.1270
46	.6947	.2238	-.2040	-.4158	-.3568	-.1079	.1666
47	.6820	.1977	-.2300	-.4252	-.3350	-.0645	.2054
48	.6691	.1716	-.2547	-.4270	-.3105	-.0251	.2349
49	.6561	.1456	-.2781	-.4286	-.2836	+.0161	.2627
50	.6428	.1198	-.3002	-.4275	-.2545	+.0563	.2854
51	.6293	.0941	-.3209	-.4259	-.2235	+.0954	.3031
52	.6157	.0686	-.3401	-.4178	-.1910	+.1326	.3153
53	.6018	.0433	-.3578	-.4093	-.1571	+.1677	.3221
54	.5878	.0182	-.3740	-.3984	-.1223	+.2002	.3234

Table (*continued*).

θ .	P_1 .	P_2 .	P_3 .	P_4 .	P_5 .	P_6 .	P_7 .
55°	·5736	—·0065	—·3886	—·3852	—·0868	+·2297	·3191
56	·5692	—·0310	—·4016	—·3698	—·0510	+·2559	·3095
57	·5446	—·0551	—·4131	—·3524	—·0150	+·2787	·2949
58	·5299	—·0788	—·4229	—·3331	·0206	+·2976	·2752
59	·5150	—·1021	—·4310	—·3119	·0557	+·3125	·2511
60	·5000	—·1250	—·4375	—·2891	·0898	+·3232	·2231
61	·4848	—·1474	—·4423	—·2647	·1229	+·3298	·1916
62	·4695	—·1694	—·4455	—·2390	·1545	+·3321	·1571
63	·4540	—·1908	—·4471	—·2121	·1844	+·3302	·1203
64	·4384	—·2117	—·4470	—·1841	·2123	+·3240	·0818
65	·4226	—·2321	—·4452	—·1552	·2381	+·3138	·0422
66	·4067	—·2518	—·4419	—·1256	·2615	+·2996	·0021
67	·3907	—·2710	—·4370	—·0955	·2824	+·2819	—·375
68	·3746	—·2896	—·4305	—·0650	·3005	+·2605	—·0763
69	·3584	—·3074	—·4225	—·0344	·3158	+·2361	—·1135
70	·3420	—·3245	—·4130	·0038	·3281	+·2089	—·1485
71	·3256	—·3410	—·4021	·0267	·3373	+·1786	—·1811
72	·3090	—·3568	—·3898	·0568	·3434	+·1472	—·2099
73	·2924	—·3718	—·3761	·0864	·3463	+·1144	—·2347
74	·2756	—·3860	—·3611	·1153	·3461	+·0795	—·2559
75	·2588	—·3995	—·3449	·1434	·3427	+·0431	—·2730
76	·2419	—·4112	—·3275	·1705	·3362	+·0076	—·2848
77	·2250	—·4241	—·3090	·1964	·3267	—·0284	—·2919
78	·2079	—·4352	—·2894	·2211	·3143	—·0644	—·2943
79	·1908	—·4454	—·2688	·2443	·2990	—·0989	—·2913
80	·1736	—·4548	—·2474	·2659	·2810	—·1321	—·2835
81	·1564	—·4633	—·2251	·2859	·2606	—·1635	—·2709
82	·1392	—·4709	—·2020	·3040	·2378	—·1926	—·2536
83	·1219	—·4777	—·1783	·3203	·2129	—·2193	—·2321
84	·1045	—·4836	—·1539	·3345	·1861	—·2431	—·2067
85	·0872	—·4886	—·1291	·3468	·1577	—·2638	—·1779
86	·0698	—·4927	—·1038	·3569	·1278	—·2811	—·1460
87	·0523	—·4959	—·0781	·3648	·0969	—·2947	—·1117
88	·0349	—·4982	—·0522	·3704	·0651	—·3045	—·0735
89	·0175	—·4995	—·0262	·3739	·0327	—·3105	—·0381
90	·0090	—·5000	—·0000	·3750	·0000	—·3125	·0000

N.B.—To find P_i , if θ lies between 90° and 180° , look up the value for the angle $180^\circ - \theta$, and change the sign if i be odd.

When P_1 , P_2 , &c. are plotted as radial heights and depths above and below a quadrant, the resulting curves, especially if coloured, are exceedingly interesting. A student can draw them to a useful scale in about one hour.

Example I.

The density σ of attracting matter on a spherical shell

1 centim. in radius is proportional to the square of the distance of any point from a diametral plane, being 6 per square centimetre where greatest: find the potential A inside and B outside. Taking the diametral plane as the equator and θ as the co-latitude, it is obvious that

$$\sigma = 6\mu^2.$$

The expansion of μ^2 in spherical harmonics is already given as

$$\mu^2 = \frac{1}{3} P_0 + \frac{2}{3} P_2.$$

So that we have σ in spherical harmonics,

$$\sigma = 2P_0 + 4P_2.$$

Hence, as A and B are derivable from the same surface harmonics,

$$A = A_0 P_0 + A_2 r^2 P_2,$$

$$B = \frac{B_0}{r} P_0 + \frac{B_2}{r^3} P_2;$$

where A_0, A_2, B_0, B_2 are constants to be found.

Now at the surface, that is where $r=1$, $A=B$, and we can apply this to every harmonic separately. Hence

$$A_0 = B_0, \quad A_2 = B_2.$$

Again, we know from the theory of attraction that the resultant force just outside and just inside the shell differs by the amount $4\pi\sigma$, or

$$\frac{dA}{dr} - \frac{dB}{dr} = 4\pi\sigma;$$

and this is to be applied to every harmonic separately. Thus, taking terms involving P_0 , we have

$$0 + B_0 r^{-2} = 4\pi \times 2;$$

or putting $r=1$,

$$B_0 = 8\pi.$$

Again, taking the second terms,

$$2A_2 r + 3B_2 r^{-4} = 4\pi \times 4;$$

or, as $A_2=B_2$, and putting $r=1$,

$$5B_2=16\pi, \quad \therefore B_2=\frac{16}{5}\pi.$$

Hence we have

$$\text{Inside potential } A=8\pi P_0+\frac{16}{5}\pi r^2 P_2,$$

$$\text{Outside potential } B=\frac{8\pi}{r}P_0+\frac{16}{5}\pi\frac{1}{r^3}P_2;$$

$$\text{or } \frac{A}{8\pi}=P_0+\frac{2}{5}r^2P_2=\alpha \text{ say, } \quad \dots \quad (1)$$

$$\frac{B}{8\pi}=\frac{1}{r}P_0+\frac{2}{5}\frac{1}{r^3}P_2=\beta \text{ say. } \quad \dots \quad (2)$$

As, for our purpose, the actual unit of potential is unimportant, we will use α for $\frac{A}{8\pi}$, and β for $\frac{B}{8\pi}$.

I had no notion of how the equipotential surfaces would shape, and I tried to avoid forming any such notion, as it was my object to test the usefulness of the tables in working out any new problem. The first thing that it strikes one to do in this case is to find the potential on the sphere itself. This can be done from either (1) or (2) by putting $r=1$, and then

$$\alpha=P_0+\frac{2}{5}P_2, \quad \text{or } 1+\frac{2}{5}P_2.$$

We can now find α for various values of θ , using the table. Thus when $\theta=0$, $P_2=1$, and therefore $\alpha=1.4$. Thus we have the following values:—

θ	0	15	30	45	60	75	90
P_2	1	.9	.625	.25	-.125	-.3995	-.5000
α	1.4	1.36	1.25	1.1	0.95	0.84	0.80

Next, take any value of θ , say $\theta=45^\circ$. Then from the table, $P_2=0.2500$, so that

$$\text{inside, } \alpha=1+\frac{1}{10}r^2;$$

$$\text{outside, } \beta = \frac{1}{r} + \frac{1}{10} \frac{1}{r^3}, \text{ or } \frac{10r^2 + 1}{10r^3}.$$

For any value of r less than 1 we calculate α ; for any value of r greater than 1 we calculate β .

For $\theta = 45^\circ$.

$r \dots$	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
$\alpha \dots$	1	1·001	1·004	1·009	1·016	1·025	1·036	1·049	1·064	1·081	1·100

$r \dots$	1·2	1·4	1·6	1·8	2·0	2·2	2·6	3·0	3·5	4	5
$\beta \dots$	·8912	·7507	·65	·572	·512	·2

Now on a sheet of squared paper I plotted the values of r and α or β , and so found the values of r for such particular values of α or β as seemed suitable for curve-drawing. In fact I found the values of r for $\theta = 45^\circ$ for various equipotential surfaces.

Repeating this for other values of θ and drawing radial lines on a sheet of paper, it was easy to draw the equipotential surfaces.

The figure (Pl. VI.) was obtained in this way by Mr. Joselin. It shows the equipotential surfaces from 0·5 to 1·4. These surfaces are surfaces of revolution. Of course the resultant force anywhere is inversely as the normal distance apart of the equipotential surfaces, and the direction of the force is everywhere normal to the equipotential surfaces.

Example II.

One circular spire of wire of radius a has an electric current C flowing in it. Find the electromagnetic potential everywhere.

At any point on the axis, z centimetres from the centre, the potential is

$$V = 2\pi C \left(1 - \frac{z}{\sqrt{z^2 + a^2}} \right). \quad \dots \quad (1)$$

It is always worth while at first to get an idea of the range of values of V .

$$\begin{aligned}\text{Putting } z=0, \quad V &= 2\pi C. \\ z=a, \quad V &= .293 \times 2\pi C. \\ z=\infty, \quad V &= 0.\end{aligned}$$

For the sake of ease of calculation let $2\pi C=1$, and let $a=1$ centim.

$$\text{Then} \quad V = 1 - \frac{z}{\sqrt{z^2 + 1}}.$$

We can expand this in powers both of z and of $\frac{1}{z}$, and we have either

$$V = 1 - z + \frac{1}{2}z^3 - \frac{3}{8}z^5 + \frac{5}{16}z^7 - \frac{35}{128}z^9 + \&c. \quad (2)$$

or

$$V = \frac{1}{2} \frac{1}{z^2} - \frac{3}{8} \frac{1}{z^4} + \frac{5}{16} \frac{1}{z^6} - \frac{35}{128} \frac{1}{z^8} + \&c. \quad (3)$$

Now, if we can find V as a function of r and θ which is correct along the axial line, then it must be correct everywhere. [This is Green's theorem, assuming that the axial line to infinity is a cylindric surface of no lateral dimensions.] But it is obvious that

$$V = 1 - rP_1 + \frac{1}{2}r^3P_3 - \frac{3}{8}r^5P_5 + \frac{5}{16}r^7P_7 - \frac{35}{128}r^9P_9 + \&c. \quad (4)$$

becomes (2) when $\theta=0$; and

$$V = \frac{1}{2} \frac{P_1}{r^2} - \frac{3}{8} \frac{P_3}{r^4} + \frac{5}{16} \frac{P_5}{r^6} - \frac{35}{128} \frac{P_7}{r^8} + \&c. \quad (5)$$

becomes (3) when $\theta=0$.

Hence these express the potential everywhere. The first of these is useful for calculation only when r is less than 1. The second is useful only when r is greater than 1.

And this shows a defect of the Spherical Harmonic method. For if r is nearly 1 we cannot easily calculate V from either of the series, having to use too many terms. It will, however,

be found, even in this case, that when we use the harmonics up to P_{11} for any value of θ we can plot V on squared paper from $r=0$ to $r=0.9$, and from $r=1.1$ to values of r as large as we please. If the intermediate part of the curve from $r=0.9$ to $r=1.1$ be drawn with a little judgment, it is astonishing how quickly and accurately the equipotential surfaces may be drawn. The result may be compared with the lines of force as worked out by the elliptic integral method of Sir William Thomson.

Any such Electromagnetic solution is also the solution of a Hydrodynamic problem.

The principle adopted in this example is very useful. It is this:—If the potential at any point along an axis is expressible as

$A_0 + A_1z + A_2z^2 + \&c.$, where $A_0, A_1, A_2, \&c.$ are constants, then the potential *anywhere* is

$$A_0P_0 + A_1P_1r + A_2P_2r^2 + \&c.$$

If the potential along an axis is

$$\frac{B_0}{z} + \frac{B_1}{z^2} + \frac{B_2}{z^3} + \&c., \text{ where } B_0, B_1, B_2, \&c. \text{ are constants,}$$

the potential *anywhere* is

$$\frac{B_0P_0}{r} + \frac{B_1P_1}{r^2} + \frac{B_2P_2}{r^3} + \&c.$$

I gave to Mr. Holland, as an example, the case of a hollow cylindric coil of wire, 2 centim. long, 1 centim. in radius inside, and 2 centim. radius outside: to find the magnetic potential everywhere when there are n turns of wire in the coil per unit length of the coil and there is unit current in the wire. Mascart gives for such a case the force at any point of the axis, and it is possible to expand Mascart's expression in powers of z and again in powers of $\frac{1}{z}$; so that it

is easy to get the potential in powers of z and of $\frac{1}{z}$, and therefore, as in the last case, the potential everywhere. Now inside the coil and well outside it only a few terms of the

series need calculation; but just at the ends of the coil the calculation is troublesome because many terms are required.

Mr. Holland tried to shorten the work in the following way. He found that the potential at points along the axis could be expressed *approximately* from $z=0$ to $z=2.5$, with a maximum error anywhere of only 2 per cent., by

$$V = 2\pi n^2 (2 - .4884z + .05513z^3 - .00518z^5 + .00022z^7),$$

z being measured from the middle point of the coil.

Hence, he said, the magnetic potential at a point r, θ is

$$V = 2\pi n^2 \{ 2 - .4884 P_1 r + .05513 P_3 r^3 - .00518 P_5 r^5 + .00022 P_7 r^7 \}.$$

From this he plotted equipotential surfaces and lines of force. He found that, inside the coil, and indeed everywhere near the coil except certain critical positions, these were approximately correct. But at the flat ends close to the wire they were absurdly wrong. It is easy now to see the physical meaning of Mr. Holland's approximation, and why we cannot use this ingenious quick way of working; but we had to be taught by experience.

Example III.

A solid bounded by a surface of revolution moves axially in an infinite mass of incompressible fluid which has no other motion than this gives to it. Find the motion.

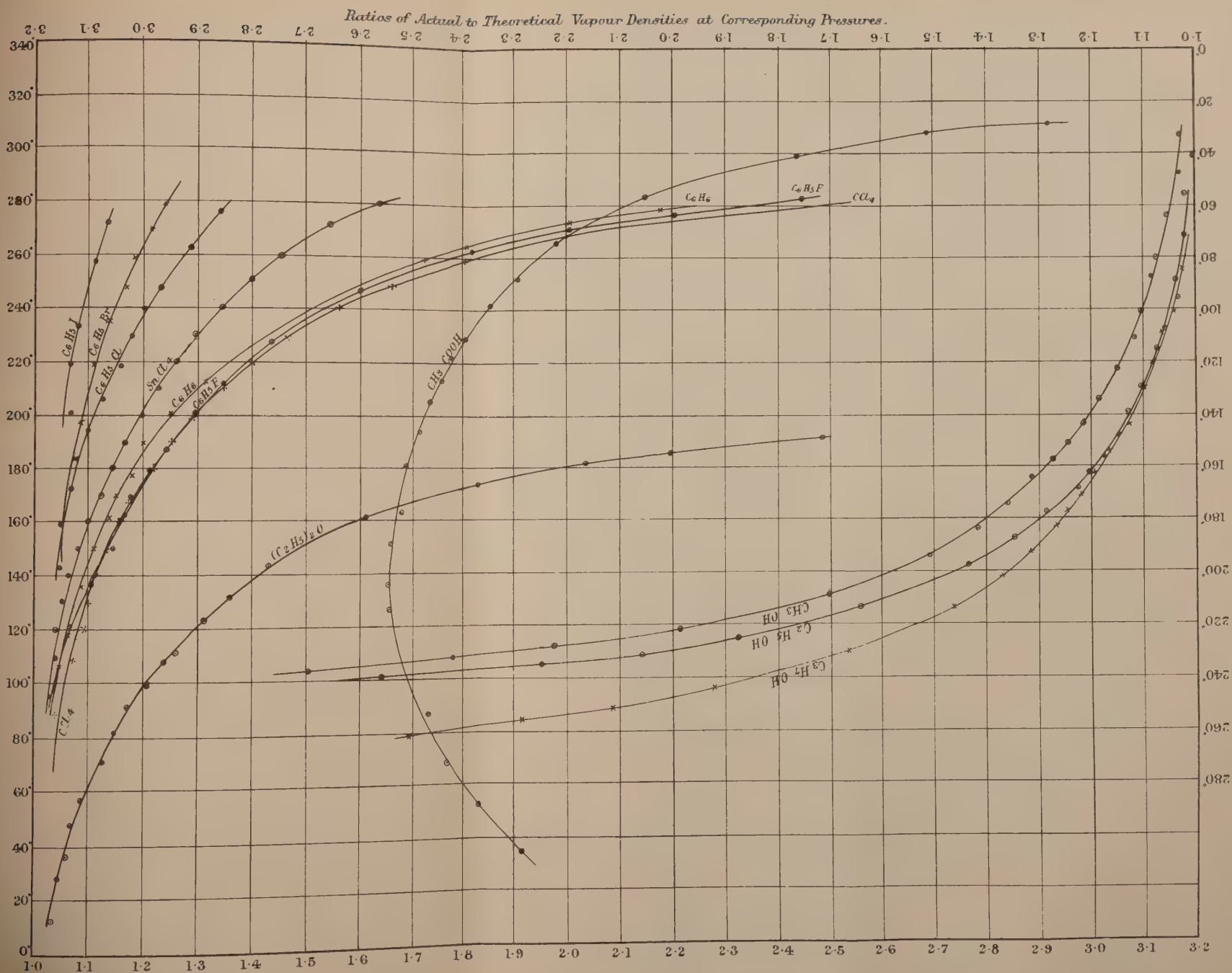
In this case

$$\nabla^2 V = 0 :$$

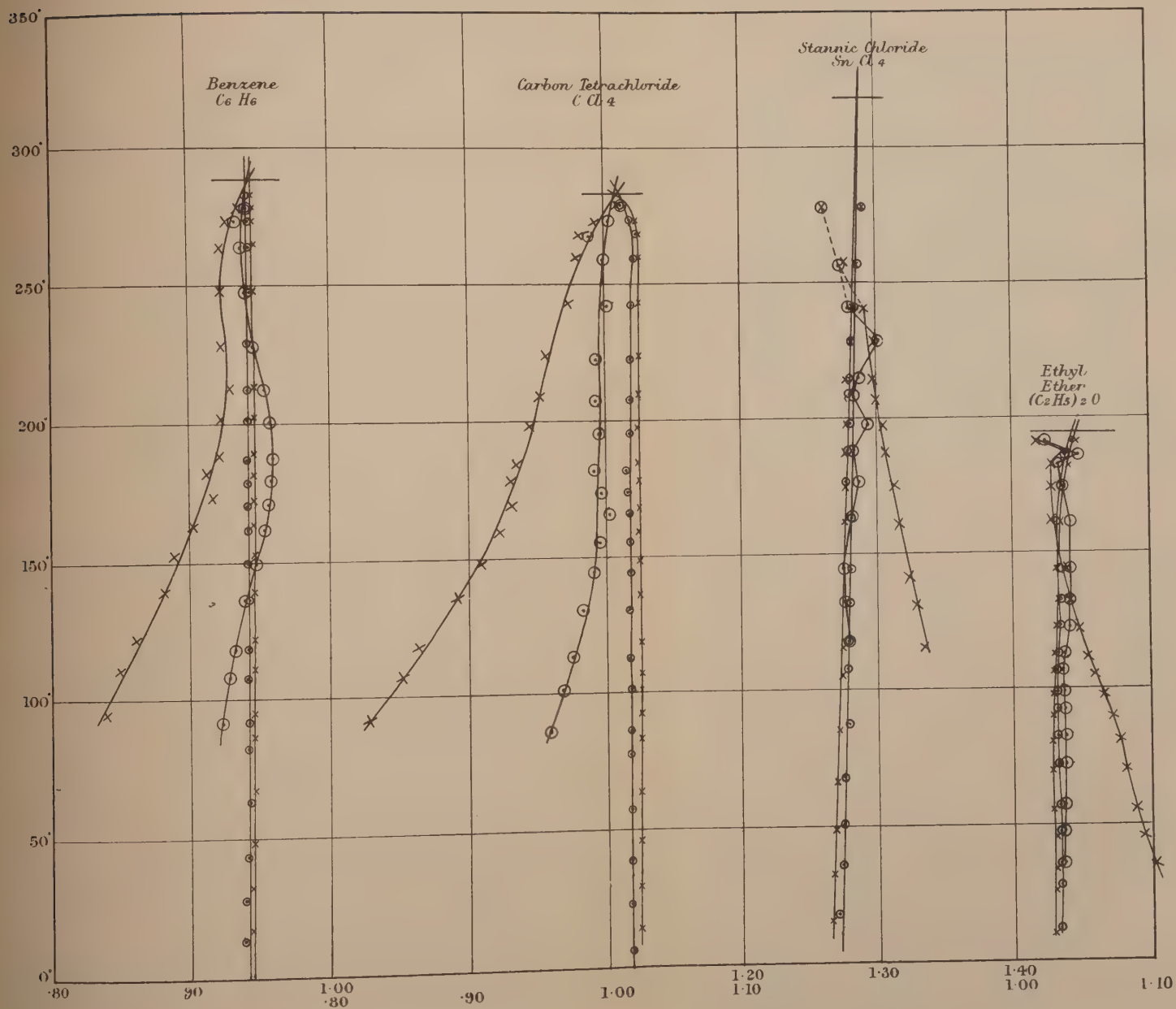
and $-\frac{dV}{dn}$ at any point at the surface of the solid (dn being an element of the normal) in the fluid is really the normal velocity of the solid itself. Again, V must be constant at any infinite distance.

This may easily be worked out for any surface of revolution. Applying it to a sphere of 1 centim. radius, let v be the velocity of the centre of the sphere in the direction parallel to the axis of z . Then $\frac{dV}{dr} = v \cos \theta$ at any point at the surface of the sphere.

Expressing V in zonal harmonics, and taking it as 0 at an infinite distance,



Group II. Fig. 4.



Group I. Fig. 3.

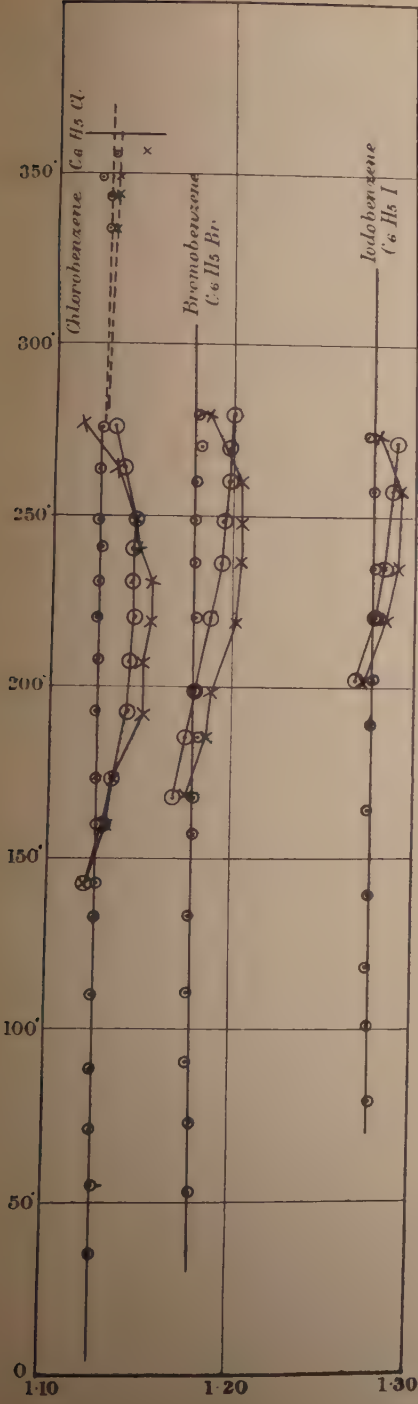
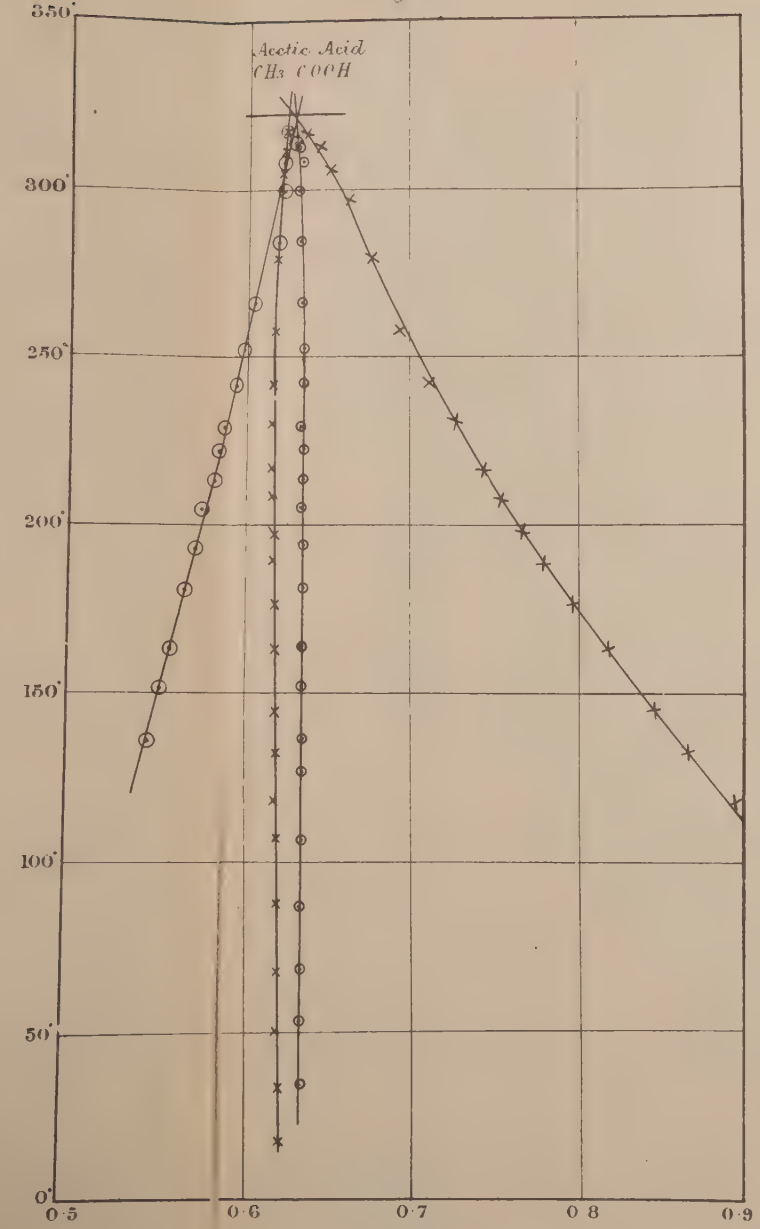
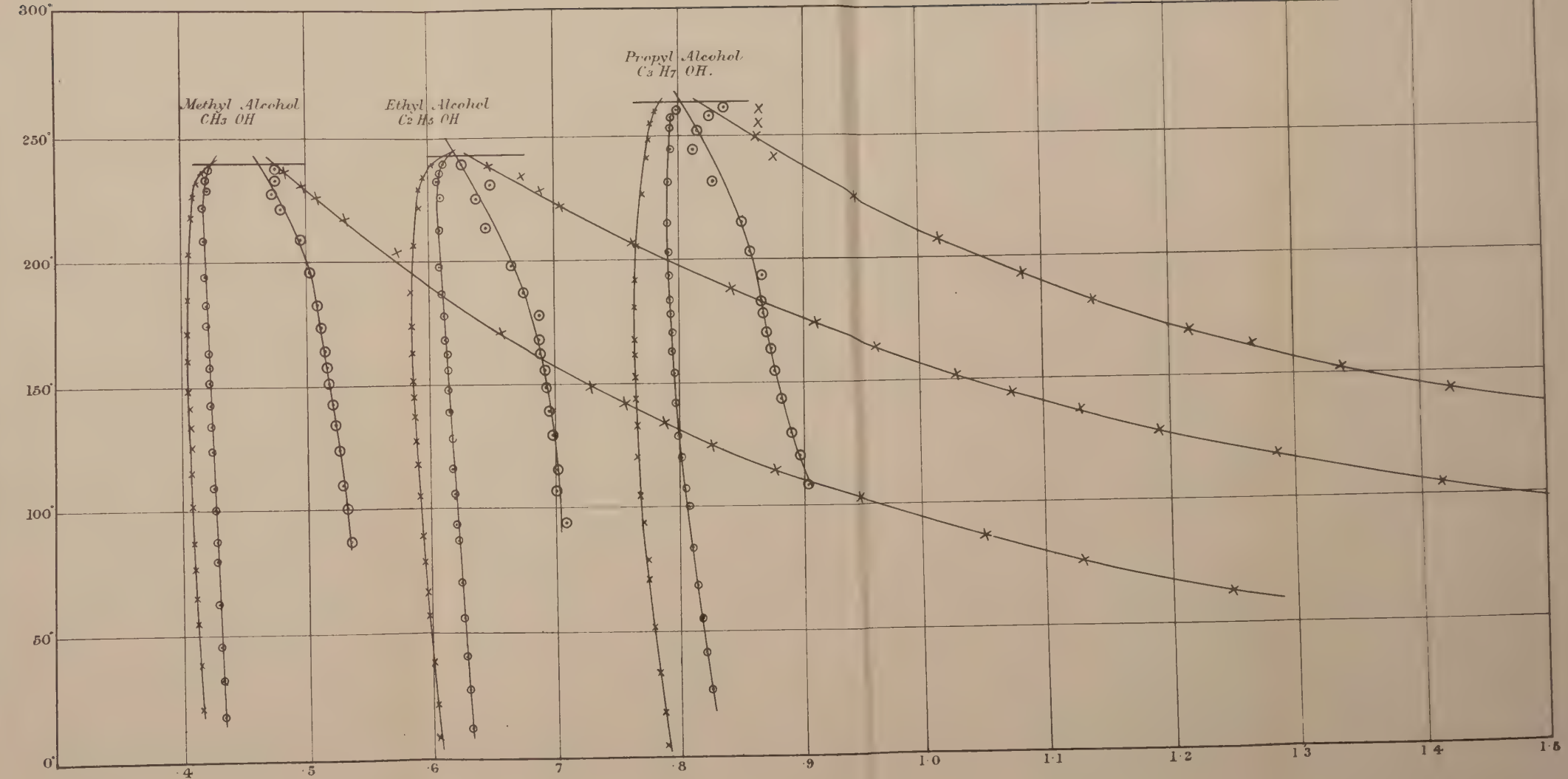


Fig. 6.



Group III.

Fig. 5.



$$V = \frac{A_0 P_0}{r} + \frac{A_1 P_1}{r^2} + \frac{A_2 P_2}{r} + \&c.$$

$\frac{dV}{dr}$ when $r=1$ is

$$v \cos \theta = -A_0 P_0 - 2A_1 P_1 - 3A_2 P_2 - \&c.$$

But $P_1 = \mu = \cos \theta$, so that the other coefficients vanish, and

$$A = -\frac{v}{2},$$

so that

$$V = -\frac{1}{2} \frac{v}{r^2} P_1. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equipotential surfaces which a student will draw from (1) are perfectly well known. It is a good exercise to draw them.

If now a velocity $-v$ is impressed upon the whole system, sphere and fluid, we have the case of the sphere at rest and the fluid moving past it. We now merely add the term $-vr \cos \theta$ to (1) and obtain

$$V = -v \left(r + \frac{1}{2r^2} \right) P_1, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which again is easily represented upon paper in equipotential surfaces.

XXVIII. *On the Generalizations of Van der Waals regarding "Corresponding" Temperatures, Pressures, and Volumes.*
By SYDNEY YOUNG, D.Sc., Professor of Chemistry, University College, Bristol*.

[Plates VII.-IX.]

IN his dissertation *Die Continuität des gasförmigen und flüssigen Zustandes* (Roth's translation, Leipzig, 1881, p. 128) Van der Waals has deduced the following generalizations from his fundamental equation

$$\left(p + \frac{a}{v^2} \right) (v - b) = R(1 + \alpha t).$$

If the absolute temperatures of various substances be proportional to their absolute critical temperatures, their vapour-

* Read November 6, 1891.

pressures will be proportional to their critical pressures, and their volumes, both as liquid and as saturated vapour, will be proportional to their critical volumes.

At the time, however, that Van der Waals's great work was published, the available experimental data were insufficient to satisfactorily test the accuracy of these generalizations.

Since then the conclusions of Van der Waals have been subjected to a large amount of criticism, both adverse and favourable; and while it is now generally recognized that the relations between the temperatures, pressures, and volumes of liquids and gases cannot be represented by so simple a formula as that quoted above, it has nevertheless been accepted as correct by some authors, who have made it the basis of further generalizations.

And although the general verdict with regard to the strict accuracy of the formula itself can hardly be considered favourable, the notion of "corresponding" states has received wide acceptance; and, indeed, the generalizations regarding corresponding temperatures, pressures, and volumes might still be true even though the formula on which they were originally based required some alteration.

In order to study the relations, for instance, between the specific volumes of different substances, determinations were made in the first place at the same temperature, generally at 0°C .; later on it was considered that the conditions would be more uniform if the comparison were made at the boiling-points of the substances under normal atmospheric pressure. It is now, however, usually admitted that in order to obtain the best results the volumes should be determined at "corresponding" temperatures—that is to say, at absolute temperatures proportional to the absolute critical temperatures of the various bodies—or at their boiling-points under corresponding pressures—the two methods of comparison being, according to Van der Waals, identical.

During the last four years I have been engaged in a research on the vapour-pressures and specific volumes—both in the liquid state and as saturated vapour—of the following substances:—benzene, fluorbenzene, chlorobenzene, bromobenzene, iodobenzene*, carbon tetrachloride and stannic chloride†;

* Trans. Chem. Soc. lv. p. 486, and lix. p. 125. † Ibid. lix. p. 911.

the vapour-pressures and specific volumes of methyl*, ethyl†, and propyl alcohols‡, and of ethyl ether§ have also been determined by Dr. Ramsay and myself. We obtained the same constants for acetic acid|| up to 280°; and I have recently extended the observations with this substance up to the critical point (*ibid.* lix. p. 903).

The methods employed for the determination of the vapour-pressures of benzene and its halogen derivatives and for the specific volumes of these bodies in the liquid state were, with slight modifications, the same as those made use of by Ramsay and myself in our researches on the alcohols and ether.

Carbon tetrachloride, however, acts on mercury at high temperatures, and stannic chloride renders it unfit for use—though the chemical action is very slight—even at low temperatures; a considerable alteration in the method of determining vapour-pressures was therefore made (*Trans. Chem. Soc.* lix. p. 917).

For the determination of the specific volumes of stannic chloride in the liquid state and of all the substances in the condition of saturated vapour an entirely different method was employed, and this has been fully described in the *Trans. Chem. Soc.* 1891, p. 37. This method is applicable to substances that attack mercury, and from the data it affords the specific volumes both of liquid and saturated vapour may be calculated; it is also available at any temperature up to the critical point. A modification of the method is described in the *Trans. Chem. Soc.* lix. p. 929.

The object of this paper is to show how far the generalizations of Van der Waals have been verified by the experimental results.

If all the relations were strictly true, it would obviously make no difference whether the specific (or molecular) volumes were compared at corresponding temperatures or corresponding pressures; but it may be stated at once that it is only in a limited number of cases that, when the absolute temperatures are proportional to the absolute critical temperatures, the

* *Phil. Trans.* clxxviii. A, p. 313.

† *Ibid.* clxxx. p. 137.

|| *Trans. Chem. Soc.* xlix. p. 790.

† *Ibid.* 1886, part i. p. 123.

§ *Ibid.* clxxviii. A, p. 57.

vapour-pressures are also proportional to the critical pressures. It is therefore necessary to compare the various substances, not only at corresponding temperatures but also at corresponding pressures. The best mode of procedure would probably be to give the temperatures, pressures, and volumes of each substance in terms of the critical constants of that body ; but the critical volumes of only a few of the compounds have been directly determined, and it has therefore been necessary to compare the various substances with one of them taken as a standard. The very simple relations observed between the four monohalogen derivatives of benzene (*Trans. Chem. Soc.* 1889, p. 486), and the fact that the constants of fluorbenzene have been determined up to the critical point, render that substance the most suitable for the purpose.

The experimental data and the ratios calculated from them are given in the following tables.

- I. Critical constants—temperature, pressure, volume.
- II. Corresponding pressures.
- III. Corresponding temperatures.
- IV. Boiling-points on absolute scale at corresponding pressures.
- V. Vapour-pressures at corresponding temperatures.
- VI. Molecular volumes of liquids at corresponding pressures.
- VII. Molecular volumes of liquids at corresponding temperatures.
- VIII. Molecular volumes of saturated vapour at corresponding pressures.
- IX. Molecular volumes of saturated vapour at corresponding temperatures.
- X. Ratios of absolute temperatures to those of fluorbenzene at corresponding pressures.
- XI. Ratios of vapour-pressures at corresponding temperatures.
- XII. Ratios of molecular volumes of liquid at corresponding pressures.
- XIII. Ratios of molecular volumes of liquid at corresponding temperatures.
- XIV. Ratios of molecular volumes of saturated vapour at corresponding pressures.

XV. Ratios of molecular volumes of saturated vapour at corresponding temperatures.

In calculating the molecular volumes the following molecular weights have been employed:—

C_6H_5F ... 95.8	C_6H_5I ... 203.4	$SnCl_4$ 259.3	C_2H_5OH ... 45.90
C_6H_5Cl ... 112.2	C_6H_6 ... 77.84	$(C_2H_5)_2O$... 73.84	C_3H_7OH ... 59.87
C_6H_5Br ... 156.6	CCl_4 ... 153.45	CH_3OH ... 31.93	CH_3COOH 59.86

TABLE I.—Critical Constants.

Substance.	Formula.	Temperature.		Pressure, in millim.	Volume, in c.c.	
		Centi- grade.	Absolute.		of a gram.	Molecular.
Fluorbenzene.....	C_6H_5F .	286.55	559.55	33912	2.43	233
Chlorobenzene	C_6H_5Cl .	(360)	(633)	(33912)	(2.34)	(262)
Bromobenzene	C_6H_5Br .	(397)	(670)	(33912)	(1.76)	(275)
Iodobenzene	C_6H_5I .	(448)	(721)	(33912)	(1.47)	(298)
Benzene	C_6H_6 .	288.5	561.5	36395	2.82	219
Carbon tetrachloride	CCl_4 .	283.15	556.15	34180		
Stannic Chloride ...	$SnCl_4$.	318.7	591.7	28080		
Ether	$(C_2H_5)_2O$.	194.4	467.4	27060		
Methyl Alcohol.....	CH_3OH .	240.0	513.0	59760		
Ethyl Alcohol	C_2H_5OH .	243.1	516.1	47850		
Propyl Alcohol	C_3H_7OH .	263.7	536.7	38120		
Acetic Acid	CH_3COOH .	321.6	594.6	43400	2.46	147

The brackets indicate calculated values. In the case of chlorobenzene, bromobenzene, and iodobenzene the critical temperatures and volumes given depend on the assumption that the critical pressures are equal. The critical constants of chlorobenzene have been observed, but the determinations could not be made with the same degree of accuracy as those of fluorobenzene, and the calculated values have therefore been adopted in this paper. The observed values are:—temperature, 359.2 to 359.45; pressure, 33926 to 33998 millim.; molecular volume, 262 to 275 cubic centim. The temperatures given in the original paper (Trans. Chem. Soc. lv. p. 518) are 360.55–360.8; but the boiling-point of mercury, the vapour of which was employed as a jacket, has since been shown to be lower than was at that time adopted.

TABLE II.
Corresponding Pressures.

Halogen Derivatives of Benzene.	$C_6H_5\cdot$	CCl_4	$SnCl_4$	$(C_2H_5)_2O$	CH_3OH	C_2H_5OH	C_3H_7OH	C_3H_7OH	CH_3COOH
20	21.46	20.16	16.56	15.96	35.24	28.22	22.48	22.48	25.60
50	53.68	50.40	41.40	39.90	88.11	70.55	56.21	56.21	63.99
100	107.3	100.8	82.24	79.80	176.2	141.1	112.4	112.4	128.0
200	214.6	201.6	165.6	159.6	352.4	282.2	224.8	224.8	256.0
400	429.3	403.2	331.2	319.2	704.9	564.4	449.7	449.7	511.9
760	815.6	766.0	629.4	606.5	1339	1072	854.3	854.3	972.6
1000	1073.0	1008	822.4	798.0	1762	1411	1124	1124	1280
1500	1610	1512	1242	1197	2643	2116	1686	1686	1920
2000	2146	2016	1656	1596	3524	2822	2248	2248	2560
3000	3220	3023	2484	2394	5287	4233	3372	3372	3839
4000	4293	4032	3312	3192	7049	5644	4497	4497	5119
5000	5366	5040	4140	3990	8811	7055	5621	5621	6390
6000	6439	6048	4920	4787	10573	8467	6745	6745	7679
7000	7513	7055	5796	5586	12337	9876	7869	7869	8959
8000	8586	8083	6624	6384	14096	11288	8993	8993	10238
10000	10732	10079	8224	7980	17624	14109	11241	11241	12798
12000	12879	12065	9636	9576	21145	16932	13490	13490	15357
15000	16098	15118	12419	11970	26430	21164	16862	16862	19197
20000	21464	20156	16562	15862	35245	28223	22480	22480	25596
25000	26831	25300	20701	19948	44055	35278	28100	28100	31995
28000	30047	28223	23185	22341	49340	39509	31477	31477	35834
30000	32197	30234	24843	23939	52869	42325	33721	33721	38394
32000	34340	32255	26497	25533	56380	45154	35975	35975	40953
33012	36395	34182	28080	27060	59760	47850	38120	38120	43400

TABLE III.
Corresponding Temperatures (absolute).

Pressure $C_6H_5F.$	$C_6H_5F.$	$C_6H_5Cl.$	$C_6H_5Br.$	$C_6H_5I.$	$C_6H_6.$	$CCl_4.$	$SnCl_4.$	$(O_2H_2)_2O.$	$CH_3OH.$	$C_2H_5OH.$	$C_3H_7OH.$	$CH_3COOH.$
20	272.25	308.0	326.0	350.8	273.2	270.6	287.9	227.4	249.6	251.1	261.15	289.3
50	289.3	327.3	346.4	372.8	290.3	287.55	305.9	241.65	265.25	266.85	277.5	307.4
100	303.9	343.8	363.9	391.6	304.95	302.05	321.35	253.85	278.6	280.3	291.5	322.95
200	320.25	362.3	383.45	412.65	321.35	318.3	338.65	267.5	293.6	295.4	307.15	340.3
400	338.75	383.2	405.6	436.5	339.95	336.7	358.2	282.95	310.55	312.45	324.9	359.95
760	358.1	405.1	428.8	461.4	359.35	355.9	378.65	299.15	328.3	330.3	343.45	380.5
1000	367.3	415.5	439.8	473.3	368.6	365.05	388.4	306.8	336.75	338.75	352.3	390.3
1500	382.0	432.15	457.4	492.2	383.35	379.65	403.95	319.1	350.2	352.35	366.4	405.9
2000	393.25	444.85	470.85	506.7	394.6	390.85	415.85	328.5	360.55	362.7	377.2	417.9
3000	410.4	464.25	491.4	528.8	411.85	407.9	448.15	342.8	376.25	378.55	393.65	436.1
4000	423.8	479.4	507.45	546.1	425.3	421.2	448.15	354.0	388.55	390.9	406.5	450.35
5000	434.85	491.95	520.7	561.1	436.35	432.2	459.85	363.25	398.7	401.1	417.1	462.1
6000	444.25	502.55	531.95	574.1	445.8	441.55	469.8	371.1	407.3	409.75	426.1	472.1
7000	452.8	512.25	542.2	584.1	454.4	450.05	478.8	378.25	415.15	417.65	434.3	481.15
8000	460.4	520.85	551.3	594.1	462.0	457.6	486.85	384.6	422.1	424.65	441.6	489.25
10000	473.6	535.75	561.3	604.1	475.25	470.7	500.8	395.6	434.2	436.8	454.25	503.25
12000	484.95	548.6	574.1	614.1	486.65	482.0	512.8	405.1	444.6	447.3	465.15	515.3
15000	499.7	565.3	589.1	629.1	501.45	496.65	528.4	417.4	458.15	460.9	479.3	531.0
20000	519.7	587.9	606.35	644.1	521.5	516.55	549.55	434.1	476.45	479.35	498.45	552.25
25000	536.0	606.35	624.5	659.1	537.85	532.75	566.8	447.7	491.4	494.35	514.1	569.55
28000	544.5	615.95	633.1	667.1	546.4	541.2	575.8	454.85	499.2	502.2	522.25	578.6
30000	550.0	622.2	638.1	672.1	551.9	546.65	581.6	459.4	504.25	507.3	527.55	584.45
32000	555.0	627.85	642.1	676.1	556.95	551.6	586.9	463.6	508.85	511.9	532.3	589.75
33912	559.55	(633.0)	(670.0)	(721.0)	561.5	556.15	591.7	467.4	513.0	516.1	536.7	594.6

TABLE IV.
Boiling-points on Absolute Scale at corresponding Pressures.

Pressure $C_6H_5F.$	$C_6H_5F.$	$C_6H_5Cl.$	$C_6H_5Br.$	$C_6H_5I.$	$C_6H_6.$	$CCl_4.$	$SnCl_4.$	$(C_2H_5)_2O.$	$CH_3OH.$	$C_2H_5OH.$	$C_3H_7OH.$	$CF_3COOP.$
20	272.25	308.05	326.3	351.25	269.25	264.05	290.75	275.7	285.7	299.4	307.05
50	289.3	327.1	346.3	372.75	286.25	281.05	308.65	291.65	301.2	314.55	325.7
100	303.9	343.4	363.6	391.3	300.9	295.5	323.7	304.85	314.05	327.45	341.5
200	320.25	361.95	383.05	412.2	317.15	311.9	341.0	270.0	319.55	328.3	341.6	359.2
400	338.75	382.8	405.3	436.1	336.3	330.5	360.3	285.0	336.0	344.0	357.25	378.9
700	358.1	404.9	429.0	461.55	355.65	350.0	380.75	301.2	353.05	360.3	373.55	399.3
1000	367.3	415.4	440.05	473.5	364.95	359.25	390.05	308.7	361.0	367.75	381.25	408.9
1500	382.0	432.2	457.8	492.45	379.85	374.15	405.6	320.9	373.3	379.6	393.2	424.1
2000	393.25	443.0	471.3	507.0	391.3	385.7	417.25	330.2	381.6	388.7	402.25	435.8
3000	410.4	464.6	492.0	529.0	408.9	403.25	435.3	344.5	396.75	402.25	415.2	453.3
4000	423.8	479.9	507.95	545.65	422.5	416.8	449.2	355.55	407.6	412.7	425.95	466.6
5000	434.85	492.45	521.05	433.5	428.05	470.3	364.5	416.5	421.2	435.7	477.5
6000	444.25	503.0	532.1	443.75	437.75	480.8	372.4	424.0	428.5	443.25	486.7
7000	452.8	512.5	542.05	452.15	446.3	479.6	379.2	430.7	435.0	449.95	494.8
8000	460.4	520.95	550.9	459.8	454.15	487.35	385.35	436.7	440.95	456.0	502.05
10000	473.6	535.5	473.3	467.8	500.75	396.0	446.75	450.95	466.4	514.4
12000	484.95	548.2	485.0	479.5	513.05	405.25	455.5	459.6	475.6	525.05
15000	499.7	564.6	500.05	494.35	528.6	417.05	466.75	470.65	487.4	538.5
20000	519.7	587.25	520.5	515.0	549.5	433.6	481.25	485.75	503.6	557.0
25000	536.0	605.7	537.1	531.95	566.7	447.45	494.7	498.15	517.2	572.35
28000	544.5	615.25	546.05	540.85	575.85	454.7	501.45	504.5	524.4	580.5
30000	550.0	621.5	551.6	546.25	581.55	459.2	505.95	508.65	529.0	585.3
32000	555.0	627.2	556.75	551.5	586.75	463.45	509.35	512.55	533.15	590.2
32912	559.55	632.5	(670.0)	(721.0)	561.5	556.15	591.7	467.4	513.0	516.1	536.7	594.6

TABLE V.
Vapour-pressures at corresponding Temperatures.

Temp. C_6H_5F .	C_6H_5F .	C_6H_5Cl .	C_6H_5Br .	C_6H_5I .	C_6H_6 .	$COCl_4$.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH .	C_2H_5OH .	C_3H_7OH .	CH_3COOH .
272.25	20	19.95	19.75	19.70	26.76	29.21	14.13	9.53
289.3	50	50.40	50.10	50.10	65.46	69.50	36.35	17.95	4.95	24.77
303.9	100	101.7	101.2	101.0	128.8	134.6	74.13	41.69	20.13	13.34	56.23
320.25	200	203.0	202.1	202.5	252.9	259.4	151.5	140.9	98.63	50.64	33.48	122.3
338.75	400	405.5	403.2	402.7	495.5	501.2	308.7	291.1	231.7	129.1	98.74	263.0
358.1	760	763.8	758.6	758.6	912.0	916.2	591.6	562.3	515.2	310.5	245.5	538.3
367.3	1000	1001	994.3	997.7	1193	1189	784.3	746.4	726.9	452.9	364.8	739.6
382.0	1500	1498	1491	1495	1766	1740	1191	1130	1211	790.7	651.6	1176
393.25	2000	1995	1982	1989	2325	2272	1598	1515	1175	973.9	1635
410.4	3000	2979	2965	2992	3432	3350	2415	2283	2897	2042	1718	2576
423.8	4000	3967	3967	4027	4545	4406	3240	3076	4198	3027	2550	3580
434.85	5000	4949	4972	5636	5451	4064	3868	5559	4102	3455	4636
444.25	6000	5943	5977	6722	6464	4881	4651	6998	5218	4411	5735
452.8	7000	6982	7015	7834	7516	5715	5476	8521	6449	5445	6894
460.4	8000	7998	8054	8913	8531	6569	6281	10070	7700	6479	8063
473.6	10000	10025	11079	10556	8232	7907	13551	10292	8580	10471
484.95	12000	12050	13228	12560	9897	9539	16788	13017	10952	13002
499.7	15000	15136	16425	15613	12402	12023	22284	17398	14504	16943
519.7	20000	20137	21752	20583	16558	16088	31659	25061	20359	23796
536.0	25000	25203	27071	25468	20725	20045	41543	33037	26761	30736
544.5	28000	28184	30200	28347	23174	22413	47479	37844	30374	34954
550.0	30000	30234	32322	30374	24860	23988	51761	41400	37932	37932
555.0	32000	32211	34435	32285	26516	25615	55847	44668	35481	40691
559.55	33912	(33912)	(33912)	(33912)	36395	34180	28080	27060	59760	47850	38120	43400

TABLE VI.
Molecular Volumes of Liquid at corresponding Pressures.

Pressure. $C_6H_5F.$	$C_6H_5F.$	$C_6H_5Cl.$	$C_6H_5Br.$	$C_6H_5I.$	$C_6H_6.$	$CCl_4.$	$SaCl_4.$	$(C_2H_5)_2O.$	$CH_3OH.$	$C_2H_5OH.$	$C_3H_7OH.$	$CH_3COOH.$
20	91.47	102.89	107.99	116.79	86.10	116.17	57.69	74.99	57.85
50	93.30	104.88	110.04	119.03	87.87	94.91	118.66	40.27	58.66	76.19	59.00
100	94.92	106.63	111.89	121.09	89.41	96.58	120.78	40.77	59.51	77.29	60.06
200	96.80	108.75	114.10	123.52	91.23	98.50	123.32	41.56	60.51	78.56	61.33
400	99.05	111.32	116.81	126.48	93.51	100.86	126.51	102.21	42.43	61.72	80.06	62.85
760	101.59	114.26	119.94	129.86	95.91	103.52	129.80	104.94	43.42	63.11	81.81	64.39
1000	102.90	115.75	121.47	131.56	97.12	104.87	131.52	106.31	43.92	63.80	82.70	65.46
1500	105.10	118.23	124.11	134.40	99.20	107.19	134.47	108.60	44.76	64.97	84.20	66.91
2000	107.00	120.24	126.24	136.72	100.96	109.10	136.88	110.46	45.45	65.97	85.47	68.09
3000	110.03	123.69	129.86	140.63	103.89	112.23	140.88	113.49	46.60	67.65	87.65	70.01
4000	112.64	126.65	132.90	143.73	106.35	114.86	144.09	116.08	47.56	69.10	89.50	71.61
5000	114.98	129.27	135.68	108.50	117.16	147.10	118.44	48.42	70.48	91.24	73.04
6000	117.06	131.72	138.23	110.52	119.29	149.82	120.70	49.23	71.67	92.87	74.38
7000	119.14	134.08	140.75	112.48	121.32	152.62	122.91	50.88	72.89	94.45	75.69
8000	121.19	136.33	143.18	114.36	123.38	155.24	125.12	52.36	73.97	96.03	76.94
10000	125.04	140.72	117.95	127.39	160.27	129.33	53.87	75.15	99.01	79.37
12000	128.80	145.10	121.55	131.99	165.56	133.36	55.17	76.33	101.99	81.79
15000	134.64	127.12	137.34	173.37	139.23	56.17	78.33	106.57	85.80
20000	145.08	137.20	148.19	187.29	149.82	60.54	88.22	114.89	91.85
25000	158.40	179.2	149.80	161.89	164.15	66.10	96.34	125.67	100.1
28000	169.35	191.5	160.20	173.55	176.18	70.92	102.59	134.41	107.4
30000	179.40	202.2	169.50	183.37	186.52	75.04	109.10	142.52	113.4
32000	193.0	219.5	182.3	196.26	201.80	81.49	118.01	153.99	121.8
33912	233.0	219.0	147.0

Approximate.

TABLE VII.
Molecular Volumes of Liquid at corresponding Temperatures.

Temp. $C_6H_6F.$	$C_6H_5F.$	$C_6H_5Cl.$	$C_6H_5Br.$	$C_6H_5I.$	$C_6H_6.$	$CCl_4.$	$SnCl_4.$	$(C_2H_5)_2O.$	$CH_3OH.$	$C_2H_5OH.$	$C_3H_7OH.$	$CH_3COOH.$
272.25	91.47	102.88	107.95	116.73	86.50	115.78	56.85
289.3	93.30	104.89	110.04	119.03	88.28	95.63	118.24	73.38	57.87
303.9	94.92	106.69	111.93	121.12	89.86	97.33	120.45	74.39	58.83
320.25	96.80	108.80	114.13	123.57	91.73	99.31	122.98	75.59	59.99
338.75	99.05	111.37	116.86	126.50	93.95	101.69	125.99	77.09	61.38
358.1	101.59	114.17	119.93	129.85	96.38	104.36	129.47	101.86	41.12	59.40	78.71	62.97
367.3	102.90	115.77	121.44	131.54	97.62	105.76	131.18	104.60	42.01	60.66	79.56	63.81
382.0	105.10	118.21	124.06	134.36	99.73	108.08	134.13	105.97	43.25	62.42	81.02	65.19
393.25	107.00	120.22	126.17	136.67	101.48	109.99	136.55	108.26	43.89	63.33	82.22	66.30
410.4	110.03	123.63	129.76	140.59	104.41	113.11	140.57	113.10	44.97	64.86	84.29	68.12
423.8	112.64	126.56	132.79	144.05	106.90	115.73	143.83	115.72	45.92	66.23	86.11	69.67
434.85	114.98	129.17	135.60	109.04	118.03	146.87	118.11	46.77	67.50	87.80	71.05
444.25	117.06	131.60	138.18	111.05	120.15	149.67	120.32	47.54	68.68	89.42	72.32
452.8	119.14	134.05	140.77	113.05	122.28	152.39	122.61	48.29	69.86	91.00	73.66
460.4	121.19	136.33	143.30	114.90	124.31	155.09	124.86	49.06	71.03	92.50	74.79
473.6	125.04	140.79	118.51	128.32	160.32	129.18	50.53	73.21	95.61	77.16
484.95	128.80	145.29	122.07	132.28	165.43	133.24	52.03	75.37	98.67	79.57
499.7	134.64	127.68	138.29	173.24	139.02	54.36	78.70	103.37	83.31
519.7	145.08	137.81	149.16	187.34	150.26	58.72	85.01	111.87	90.00
536.0	158.40	180.0	150.60	162.81	164.37	64.31	93.52	122.55	98.25
544.5	169.35	192.4	160.70	174.09	176.48	69.13	99.92	131.41	105.45
550.0	179.40	204.0	170.00	184.06	187.11	73.63	106.58	139.50	112.15
555.0	193.0	222.4	183.00	196.57	202.32	80.43	116.13	150.87	120.8
559.55	233.0	219.0	147

Approximate.

TABLE IX.
Molecular Volumes of Saturated Vapour at corresponding Temperatures.

Temp. C_6H_5F	C_6H_5F	C_6H_5Cl	C_6H_5Br	C_6H_5I	C_6H_6	CCl_4	$SnCl_4$	$(C_2H_5)_2O$	CH_3OH	C_2H_5OH	C_3H_7OH	CH_3COOH
272.25	384500
289.3	193500
303.9	99700
320.25	49500
338.75	26150
358.1	19680
367.3	22000	24650	25850	28000	18550	18250	38350	58600	81300	150500	12940
382.0	15000	16920	17770	19280	12780	12760	29400	31650	38000	65300	9625
393.25	11400	12930	13540	14710	9865	9865	19910	16400	27400	45700	6280
410.4	7780	8830	9225	9930	6800	6870	15050	12400	11950	18600	4613
423.8	5785	6653	6958	7413	5164	5266	10100	8320	7260	10850	3622
434.85	4634	5353	5576	4198	4281	7595	6237	5065	7430	2968
444.25	3857	4447	4645	3544	3597	6060	4972	3828	5521	2495
452.8	3298	3780	3954	3023	3083	5035	4116	3041	4345	2138
460.4	2871	3281	3414	2649	2692	4285	3491	2489	3532	1652
473.6	2265	2567	2099	3733	3027	2094	2951	1326
484.95	1862	2077	1734	1776	2944	2374	1560	2178	1005
499.7	1447	1340	1388	2404	1939	1219	1690	687
519.7	1009	934	986	1845	1505	890	1218	487
536.0	733	677	719	1274	1040	578	770	393
544.5	601	569	592	753	389	516	335
550.0	516	485	514	619	305	415	281
555.0	440	446	538	257	349	381
559.55	233	219	451	213	285	147

TABLE XI.
Ratios of Vapour-Pressures to those of Fluorbenzene at corresponding Temperatures.

Temp. $\text{O}_6\text{H}_5\text{F.}$	$\text{C}_6\text{H}_5\text{F.}$	$\text{C}_6\text{H}_5\text{Cl.}$	$\text{C}_6\text{H}_5\text{Br.}$	$\text{C}_6\text{H}_5\text{I.}$	$\text{C}_6\text{H}_6.$	$\text{CCl}_4.$	$\text{SnCl}_4.$	$(\text{C}_2\text{H}_5)_2\text{O.}$	$\text{OH}_3\text{OH. C}_2\text{H}_5\text{OH.}$	$\text{C}_2\text{H}_5\text{OH.}$	$\text{CH}_3\text{COOH.}$
272.25	1.0000	.998	.987	.985	1.338	1.460	.706476
289.3	"	1.008	1.012	1.002	1.309	1.390	.727090	.495
303.9	"	1.016	1.012	1.010	1.288	1.346	.741201	.133	.562
320.25	"	1.015	1.010	1.013	1.265	1.297	.758	.705	.253	.182	.612
338.75	"	1.014	1.008	1.007	1.239	1.253	.772	.728	.323	.247	.658
358.1	"	1.005	.998	.998	1.200	1.206	.778	.740	.409	.323	.708
367.3	"	1.001	.994	.998	1.193	1.189	.784	.746	.453	.365	.740
382.0	"	.999	.994	.996	1.177	1.160	.794	.753	.527	.434	.784
393.25	"	.998	.991	.994	1.163	1.136	.799	.758487	.817
410.4	"	.993	.988	.997	1.144	1.117	.805	.761	.681	.573	.859
423.8	"	.992	.992	1.007	1.136	1.101	.810	.769	1.049	.757	.897
434.85	"	.990	.994	1.127	1.090	.813	.774	1.112	.820	.931
444.25	"	.991	.996	1.120	1.077	.814	.775	1.167	.870	.735
452.8	"	.997	1.002	1.119	1.074	.816	.782	1.217	.921	.778
460.4	"	1.000	1.007	1.114	1.066	.821	.785	1.259	.962	.810
473.6	"	1.002	1.108	1.056	.823	.791	1.335	.983	1.008
484.95	"	1.004	1.102	1.047	.825	.795	1.399	.913	1.047
499.7	"	1.009	1.095	1.041	.827	.801	1.483	.967	1.083
519.7	"	1.007	1.088	1.029	.828	.804	1.583	1.160	1.130
536.0	"	1.008	1.083	1.019	.829	.802	1.662	1.253	1.190
544.5	"	1.006	1.078	1.012	.828	.800	1.696	1.322	1.229
550.0	"	1.008	1.077	1.013	.829	.800	1.696	1.351	1.245
555.0	"	1.007	1.076	1.009	.829	.801	1.725	1.380	1.264
559.55	"	1.073	1.008	.828	.798	1.745	1.396	1.272
									1.762	1.411	1.280

Ratio of Highest to Lowest Ratio.

(1.026 | 1.025 | 1.028 | 1.247 | 1.448 | 1.174 | 1.140 | 4.91 | 7.02 | 11.35 | 2.68

TABLE XII.

Ratios of Molecular Volumes of Liquid to those of Fluorobenzene at corresponding Pressures.

Pressure C_6H_5F	$C_6H_5F_2$	C_6H_5Cl	C_6H_5Br	C_6H_5I	C_6H_6	CCl_4	$SnCl_4$	$(C_2H_5)_2O$	CH_3OH	C_2H_5OH	C_3H_7OH	CH_3COOH
20	1.0000	1.1248	1.1806	1.2768	.9413	1.27006307	.8198	.6324
50	"	1.1241	1.1794	1.2758	.9418	1.0173	1.27184317	.6287	.8166	.6324
100	"	1.1234	1.1788	1.2757	.9420	1.0175	1.27244295	.6269	.8143	.6327
200	"	1.1234	1.1787	1.2760	.9425	1.0176	1.27404293	.6251	.8115	.6336
400	"	1.1239	1.1793	1.2769	.9441	1.0183	1.2773	1.0319	.4294	.6232	.8083	.6345
700	"	1.1247	1.1806	1.2783	.9441	1.0190	1.2777	1.0330	.4274	.6213	.8053	.6338
1000	"	1.1249	1.1805	1.2785	.9438	1.0191	1.2781	1.0332	.4268	.6200	.8037	.6362
1500	"	1.1249	1.1809	1.2778	.9439	1.0199	1.2795	1.0333	.4259	.6182	.8012	.6366
2000	"	1.1238	1.1798	1.2778	.9436	1.0197	1.2793	1.0324	.4247	.6166	.7987	.6344
3000	"	1.1241	1.1802	1.2781	.9442	1.0200	1.2804	1.0315	.4235	.6148	.7966	.6363
4000	"	1.1244	1.1799	1.2760	.9442	1.0197	1.2792	1.0305	.4222	.6137	.7953	.6357
5000	"	1.1243	1.18009436	1.0190	1.2794	1.0301	.4211	.6130	.7936	.6359
6000	"	1.1252	1.18089441	1.0191	1.2799	1.0311	.4210	.6123	.7934	.6354
7000	"	1.1254	1.18149441	1.0183	1.2810	1.0317	.4202	.6118	.7927	.6353
8000	"	1.1249	1.18149436	1.0180	1.2810	1.0324	.4198	.6103	.7924	.6349
10000	"	1.12549433	1.0188	1.2818	1.0343	.4188	.6080	.7918	.6348
12000	"	1.12659437	1.0201	1.2854	1.0354	.4182	.6081	.7915	.6350
15000	"9442	1.0200	1.2877	1.0341	.4172	.6063	.7915	.6335
20000	"9457	1.0214	1.2910	1.0327	.4173	.6081	.7919	.6331
25000	"	1.13109457	1.0220	1.0364	.4173	.6082	.7933	.6319
28000	"	1.13109460	1.0248	1.0403	.4188	.6058	.7937	.6342
30000	"	1.12709448	1.0222	1.0397	.4183	.6082	.7944	.6321
32000	"	1.13509446	1.0169	1.0456	.4222	.6114	.7978	.6311
33912	"94006310
Mean Ratios.												
		* 1.1246	1.1802	1.2772	.9439	1.0195	1.2793	1.0342	.4228	.6153	.7995	.6342
Ratio of Highest to Lowest Ratio.												
		* 1.003	1.002	1.002	1.006	1.008	1.017	1.015	1.035	1.041	1.036	1.009

* Omitting approximate values.

Ratios of Molecular Volumes of Liquid to those of Fluorbenzene at corresponding Temperatures.

Temp. C_6H_5F .	C_6H_5F .	C_6H_5Cl .	C_6H_5Br .	C_6H_5I .	C_6H_6 .	CCl_4 .	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH .	C_2H_5OH .	C_3H_7OH .	CH_3COOH .
272.25	1.0000	1.1247	1.1802	1.2761	.9457	1.26576215
289.3	"	1.1242	1.1794	1.2758	.9462	1.0250	1.26737865	.6203
303.9	"	1.1240	1.1792	1.2760	.9467	1.0254	1.26896044	.7837	.6198
320.25	"	1.1239	1.1790	1.2765	.9476	1.0260	1.27054169	.6022	.7809	.6197
338.75	"	1.1244	1.1798	1.2772	.9485	1.0267	1.2720	1.0284	.4152	.5997	.7783	.6197
358.1	"	1.1238	1.1805	1.2782	.9487	1.0273	1.2744	1.0297	.4136	.5971	.7748	.6198
367.3	"	1.1250	1.1802	1.2783	.9487	1.0278	1.2748	1.0299	.4127	.5958	.7732	.6201
382.0	"	1.1248	1.1804	1.2784	.9489	1.0283	1.2763	1.0301	.4115	.5939	.7709	.6203
393.25	"	1.1236	1.1792	1.2773	.9484	1.0280	1.2761	1.0290	.4102	.5919	.7684	.6196
410.4	"	1.1236	1.1793	1.2777	.9489	1.0280	1.2775	1.0279	.4087	.5895	.7661	.6191
423.8	"	1.1236	1.1789	1.2789	.9491	1.0275	1.2769	1.0273	.4076	.5880	.7604	.6185
434.85	"	1.1234	1.18049484	1.0266	1.2773	1.0272	.4068	.5871	.7636	.6179
444.25	"	1.1242	1.18159487	1.0264	1.2786	1.0279	.4061	.5867	.7639	.6178
452.8	"	1.1252	1.18159489	1.0264	1.2791	1.0291	.4054	.5864	.7638	.6174
460.4	"	1.1249	1.18249481	1.0257	1.2797	1.0302	.4048	.5861	.7633	.6171
473.6	"	1.12609478	1.0262	1.2822	1.0331	.4041	.5855	.7646	.6171
484.95	"	1.12809477	1.0270	1.2844	1.0345	.4040	.5851	.7660	.6178
499.7	"9483	1.0271	1.2867	1.0326	.4038	.5845	.7677	.6188
519.7	"	1.13609499	1.0281	1.2913	1.0357	.4037	.5859	.7711	.6204
536.0	"	1.13609508	1.0279	1.0377	.4060	.5904	.7737	.6203
544.5	"	1.13709489	1.0280	1.0421	.4082	.5900	.7760	.6227
550.0	"	1.13709476	1.0260	1.0430	.4104	.5941	.7776	.6251
555.0	"	1.15209482	1.0188	1.0483	.4167	.6017	.7817	.6259
559.55	"94006310

Mean Ratios.

*1.1245	1.1800	1.2773	.9479	1.0286	1.277	1.033	.409	.592	.772	.623
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Ratio of Highest to Lowest Ratio.

*1.004	1.003	1.002	1.011	1.009	1.020	1.021	1.032	1.030	1.034	1.023
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* Omitting approximate values

TABLE XV.

Ratio of Molecular Volumes of Saturated Vapour to those of Fluorbenzene at corresponding Temperatures.

Temp. C_6H_5F .	C_6H_5F .	C_6H_5Cl .	C_6H_5Br .	C_6H_5I .	C_6H_6 .	CCl_4 .	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH .	C_2H_5OH .	C_3H_7OH .	CH_3COOH .		
367.3	1.000	1.118	1.174	1.272	.842	.828	1.336	1.101	1.244	2.076	2.678	.894		
382.0	"	1.130	1.186	1.286	.853	.852	1.329	1.092	1.126	1.805	2.254	.864		
393.25	"	1.135	1.189	1.292	.866	.866	1.322	1.088	1.049	1.635	2.007	.845		
410.4	"	1.150	1.201	1.293	.885	.895	1.316	1.083	.945	1.414	1.714	.818		
423.8	"	1.150	1.203	1.281	.893	.910	1.313	1.078	.875	1.284	1.541	.797		
434.85	"	1.156	1.203906	.924	1.308	1.073	.826	1.191	1.422	.782		
444.25	"	1.155	1.204919	.933	1.305	1.067	.788	1.126	1.337	.769		
452.8	"	1.146	1.199917	.935	1.300	1.059	.755	1.071	1.266	.756		
460.4	"	1.143	1.189923	.938	1.300	1.054	.729	1.028	1.216	.745		
473.6	"	1.134927	.949	1.300	1.048	.689	.962	1.139	.729		
484.95	"	1.116931	.954	1.291	1.041	.655	.908	1.081	.712		
499.7	"926	.959	1.275	1.040	.615	.841	1.016	.694		
519.7	"926	.977	1.262	1.030	.572	.763	.943	.681		
536.0	"924	.982	1.028	.530	.705	.875	.665		
544.5	"930	.985	1.030	.508	.690	.861	.653		
550.0	"940	.995	1.041	.497	.676	.864	.649		
555.0	"	1.014	1.025	.484	.648	.866	.639		
559.55	"940631		
Mean Ratios.														
				1.139	1.194	1.285	Ratio of Highest to Lowest Ratio.							
				1.036	1.026	1.017	1.116	1.224	1.059	1.074	2.570	3.204	3.110	1.417

TABLE XVI.

Percentage-differences between Highest and Lowest Ratios in each comparison with Fluorbenzene.

	C_6H_5Cl	C_6H_5Br	C_6H_5I	C_6H_6	OCl_4	$SnCl_4$	$(C_2H_5)_2O$	CH_3OH	C_2H_5OH	C_3H_7OH	CH_3COOH
Absolute temperatures at corresponding pressures	0.2	0.2	0.2	1.4	2.5	1.0	1.1	10.5	13.8	14.6	6.1
Vapour-pressures at corresponding temperatures...	2.6	2.5	2.8	24.7	44.8	17.4	14.0	391.0	602.0	1035.0	168.5
Molecular volumes of liquid at corresponding pressures	0.3	0.2	0.2	0.6	0.8	1.7	1.5	3.5	4.1	3.6	0.9
Molecular volumes of liquid at corresponding temperatures	0.4	0.3	0.2	1.1	0.9	2.0	2.1	3.2	3.0	3.4	2.3
Molecular volumes of saturated vapour at corresponding pressures	2.2	2.7	1.7	3.9	5.9	3.2	1.0	12.9	13.0	11.5	15.8
Molecular volumes of saturated vapour at corresponding temperatures...	3.6	2.6	1.7	11.6	22.4	5.9	7.4	157.0	220.4	211.0	41.7

The degree of deviation from constancy of the various ratios is indicated by the values given at the foot of each table, representing the ratio of the highest to the lowest ratio in each vertical column. These values are collected together in Table XVI. (p. 252), but for the sake of greater clearness they are given as percentage-differences between the highest and lowest ratios in each case, the lowest ratio being always taken as 100.

It will be seen that the halogen derivatives of benzene show very much smaller deviations from constancy than the other compounds; and I have previously suggested (*Trans. Chem. Soc.* 1889, p. 486; 1891, p. 125) that the generalizations of Van der Waals do hold good for these bodies, in which case the deviations of the ratios from constancy may be taken as an approximate measure of the experimental errors to be expected with other compounds. It must, however, be admitted that the mean ratios of the absolute temperatures differ sensibly from the mean ratios of the molecular volumes of liquid, though they should be identical if Van der Waals's generalizations were strictly true, while the errors in these determinations are probably very small. Indeed, in the comparison of bromobenzene with fluorobenzene the difference between the mean ratios amounts to 1.5 per cent. (Table XXI.), while the difference between the highest and lowest ratio in either case is only 0.2 per cent. (Table XVI.). It is probable, therefore, that the generalizations of Van der Waals are not quite, though very nearly, true for these bodies.

The deviations from constancy are smallest in the comparisons of the absolute temperatures at corresponding pressures, and of the molecular volumes of liquid at corresponding pressures and temperatures; and this is no doubt to be expected, since the ratio of the highest to the lowest absolute temperature or volume of liquid for any one substance does not amount to 3 : 1, whereas the ratio of the highest to the lowest pressure is from 1700 to 10000 : 1, and of the highest to the lowest molecular volume of saturated vapour from 3.7 : 1 in the case of iodobenzene, where the range is limited, to 2670 to 1 with acetic acid. It is also impossible to determine the volumes of saturated vapour with anything like the same

degree of accuracy as the volumes of liquid, the liability to error by the new method being especially great at the lowest temperatures.

The eleven substances which are compared with fluorbenzene may be arranged in three groups as below :—

1. *Chlorobenzene, Bromobenzene, Iodobenzene.*—As already pointed out, the generalizations of Van der Waals are very nearly true for these bodies when compared with fluorbenzene.

2. *Benzene, Carbon tetrachloride, Stannic chloride, Ether.*—With these substances the generalizations may be taken as rough approximations to the truth, but the deviations of the ratios from constancy are in most cases much too large to be attributed to error of experiment.

3. *The three Alcohols and Acetic Acid.*—The majority of the generalizations do not hold good at all ; the deviations of the ratios from constancy are, however, not very great in the case of the molecular volumes of liquid at corresponding pressures and temperatures.

Discussion of the Generalizations.

1. *Corresponding Temperatures and Pressures.*—It must, I think, be concluded that the statement that “if the absolute temperatures of various substances are proportional to their absolute critical temperatures their vapour-pressures will be proportional to their critical pressures” has not been proved by experiment to be true except in a very limited number of cases; indeed, when the alcohols and acetic acid are compared with fluorbenzene, the statement is quite wide of the mark. It follows, therefore, that in the comparison of the molecular volumes the ratios at corresponding pressures must differ from those at corresponding temperatures.

2. *Molecular Volumes of Liquid.*—The deviations of the ratios from constancy are smaller in this case than in any of the others, but are certainly not within the limits of experimental error. The comparisons at corresponding pressures seem to be somewhat better on the whole than at corresponding temperatures, but the difference is not very marked.

3. *Molecular Volumes of Saturated Vapour.*—At corresponding pressures the deviations from constancy are within the limits of experimental error in the case of ether and stannic chloride, and are relatively small for benzene and carbon tetrachloride.

There can be no doubt that with the saturated vapours the comparisons at corresponding pressures are very much better than at corresponding temperatures; and it may therefore be concluded that it is better to compare the molecular volumes both of liquid and of saturated vapour at corresponding pressures than at corresponding temperatures.

Saturated Vapours.

It has been pointed out (Trans. Chem. Soc. 1891, p. 137) that if the generalizations of Van der Waals were strictly true, it would follow that the ratios of the actual densities of the saturated vapours of different substances to their theoretical densities should be equal at corresponding pressures; and as this method of comparison is a very convenient one I have thought it worth while to tabulate the ratios, although it has already been shown that the relation cannot be strictly true. The values for benzene and its halogen derivatives have already been published (*loc. cit.*); but there are one or two obvious small irregularities, and I have therefore mapped the ratios against temperature and constructed curves for each substance (Plate VII.). The smoothed values read from the curves are given in this paper, and the molecular volumes of saturated vapour have also been recalculated from the density ratios. The same method of procedure has been adopted with the other substances.

The close similarity in the behaviour of the halogen derivatives of benzene and the approximate agreement in the case of the members of Group II. is very clearly shown in Table XVII.

It is generally admitted that gaseous acetic acid at low temperatures contains molecules more complex than those corresponding to the ordinary formula $C_2H_4O_2$, and from the

TABLE XVII.

Ratios of Actual to Theoretical Densities of Saturated Vapour at corresponding Pressures.

Pressure. $C_6H_5F.$	$C_6H_5F.$	$C_6H_5Cl.$	$C_6H_5Br.$	$C_6H_5I.$	$C_6H_6.$	$CCl_4.$	$SnCl_4.$	$(O_2H_2)_2O.$	$CH_3OH.$	$C_2H_5OH.$	$C_3H_7OH.$	$CH_3COOH.$
20	1.907
50	1.829
100	1.770
200	1.718
400	1.657
760	1.040	1.046	1.071	1.033	1.047	1.654
1000	1.037	1.047	1.063	1.053	1.037	1.047	1.047	1.054	1.084	1.042	1.060	1.654
1500	1.056	1.059	1.077	1.065	1.052	1.059	1.063	1.072	1.105	1.059	1.081	1.659
2000	1.073	1.072	1.091	1.079	1.066	1.073	1.079	1.090	1.122	1.075	1.099	1.669
3000	1.107	1.099	1.116	1.107	1.091	1.098	1.107	1.122	1.157	1.103	1.132	1.690
4000	1.138	1.127	1.142	1.136	1.112	1.122	1.136	1.153	1.189	1.129	1.164	1.713
5000	1.166	1.153	1.168	1.159	1.136	1.147	1.162	1.180	1.219	1.153	1.192	1.735
6000	1.193	1.181	1.192	1.159	1.173	1.187	1.207	1.249	1.177	1.217	1.759
7000	1.217	1.206	1.217	1.180	1.197	1.213	1.233	1.277	1.203	1.242	1.781
8000	1.247	1.233	1.242	1.206	1.233	1.238	1.260	1.306	1.230	1.267	1.805
10000	1.300	1.291	1.260	1.275	1.287	1.308	1.360	1.275	1.315	1.852
12000	1.349	1.347	1.316	1.329	1.343	1.358	1.419	1.343	1.373	1.904
15000	1.431	1.408	1.410	1.435	1.436	1.507	1.437	1.462	1.983
20000	1.600	1.582	1.571	1.619	1.605	1.700	1.643	1.664	2.153
25000	1.818	2.006	1.994	1.837	1.985	1.876	1.927	2.434
28000	2.011	2.006	1.994	2.032	2.228	2.054	2.116	2.691
30000	2.206	2.180	2.164	2.199	2.420	2.250	2.282	2.920
32000	2.450	2.376	2.485	2.698	2.561	2.504	3.252
33912	4.400	(4.43)	(4.46)	(4.43)	4.370	(4.30)	(4.35)	(4.40)	(5.18)	(4.65)	(4.70)	5.792

The bracketed values are calculated from the molecular critical volumes given in Table XXII.

very high density at the critical point it would appear that many of these complex molecules have escaped dissociation even at this high temperature.

It will be seen that with the alcohols the differences from the other substances become generally more marked as the critical point is approached. At the highest pressure the vapour-density of methyl alcohol is very distinctly higher than that of any member of the first or second group at the corresponding pressure ; and this fact appears to favour the conclusion of M. Guye and others, that some of the molecules of methyl alcohol at the critical point and in the liquid state at all temperatures are more complex than in the ordinary gaseous state. On the other hand, by a comparison of the densities of the saturated vapours of acetic acid with those of the alcohols and ether, Dr. Ramsay and I were led to the conclusion that the molecules of ordinary substances, including the alcohols, are not more complex in the liquid than in the gaseous state. This conclusion, so far as the members of Groups I. and II. are concerned, is strengthened by the results given in the preceding Table ; but it is certainly weakened to some extent in the case of the alcohols, or at any rate of methyl alcohol.

There is no doubt that these bodies do show marked differences in many of their properties from the majority of compounds ; and the most plausible explanation of these differences seems to be the existence of complex molecules in the liquid state, although there is ample proof that they are not present in the saturated vapours at low temperatures.

As the alcohols differ so widely from other substances, it is of interest to find whether the relations of Van der Waals hold good when they are compared among themselves. I have therefore calculated the ratios of the absolute temperatures, pressures, and volumes of ethyl and propyl alcohol to methyl alcohol and of propyl alcohol to ethyl alcohol. The results are given in the following Tables :—

TABLE XVIII.—Alcohols at corresponding Pressures.

Pressures of C_6H_5F .	Absolute Temperatures.			Molecular Volumes of Liquid.				Molecular Volumes of Vapour.		
	$\frac{C_2H_5OH}{CH_3OH}$.	$\frac{C_3H_7OH}{CH_3OH}$.	$\frac{C_4H_9OH}{C_2H_5OH}$.	$\frac{C_2H_5OH}{CH_3OH}$.	$\frac{C_3H_7OH}{CH_3OH}$.	$\frac{C_4H_9OH}{C_2H_5OH}$.	$\frac{C_5H_{11}OH}{C_3H_7OH}$.	$\frac{C_2H_5OH}{CH_3OH}$.	$\frac{C_3H_7OH}{CH_3OH}$.	$\frac{C_4H_9OH}{C_2H_5OH}$.
20	1.0363	1.0880	1.0479	1.8918	1.2998
50	1.0327	1.0785	1.0443	1.4565	1.8960	1.2989
100	1.0302	1.0741	1.0427	1.4597	1.8960	1.2989
200	1.0274	1.0690	1.0405	1.4562	1.8903	1.2981
400	1.0238	1.0632	1.0385	1.4546	1.8866	1.2971
760	1.0205	1.0581	1.0368	1.4535	1.8840	1.2962
1000	1.0187	1.0561	1.0367	1.4527	1.8831	1.2962
1500	1.0169	1.0533	1.0358	1.4515	1.8812	1.2960
2000	1.0186	1.0541	1.0349	1.4490	1.8806	1.2960
3000	1.0139	1.0465	1.0322	1.4493	1.8765	1.2977
4000	1.0125	1.0450	1.0321	1.4534	1.8838	1.2959
5000	1.0113	1.0461	1.0344	1.4555	1.8843	1.2946
6000	1.0106	1.0454	1.0344	1.4543	1.8841	1.2958
7000	1.0100	1.0447	1.0344	1.4538	1.8874	1.2957
8000	1.0097	1.0442	1.0341	1.4538	1.8874	1.2983
10000	1.0094	1.0440	1.0343	1.4543	1.8864	1.3001
12000	1.0090	1.0441	1.0348	1.4541	1.8933	1.3021
15000	1.0084	1.0443	1.0356	1.4534	1.8974	1.3054
20000	1.0093	1.0465	1.0368	1.4572	1.8978	1.3023
25000	1.0070	1.0455	1.0382	1.4576	1.9013	1.3044
28000	1.0081	1.0458	1.0394	1.4466	1.8953	1.3102
30000	1.0061	1.0464	1.0400	1.4540	1.8994	1.3063
32000	1.0063	1.0467	1.0402	1.4482	1.8897	1.3049
33912	1.0060	1.0462	1.0399
Mean Ratios.										
	1.030	1.040	1.015	1.4534	1.8890	1.2999	1.320	1.693	1.274	
				Ratio of Highest to Lowest Ratio.						
				1.012	1.013	1.012	1.040	1.057	1.063	

TABLE XIX.—Alcohols at corresponding Temperatures.

Absolute Temperatures of C_6H_5F .	Vapour-Pressures.			Molecular Volumes of Liquid.			Molecular Volumes of Saturated Vapour.		
	C_2H_5OH CH_3OH	C_3H_7OH CH_3OH	C_3H_7OH C_2H_5OH	C_2H_5OH CH_3OH	C_3H_7OH CH_3OH	C_3H_7OH C_2H_5OH	C_2H_5OH CH_3OH	C_3H_7OH CH_3OH	C_3H_7OH C_2H_5OH
272-25
289-3	276	662	1-2968
303-9	483	320	720	1-4443	1-8730	1-2968
320-25	513	370	765	1-4444	1-8746	1-2978
338-75	557	426	791	1-4439	1-8735	1-2975	1-851
358-1	602	476	805	1-4435	1-8734	1-2978	1-718
387-3	623	502	824	1-4431	1-8734	1-2981	1-669
382-0	653	538	829	1-4430	1-8734	1-2983	1-603	2-153	1-290
393-25	841	1-4430	1-8734	1-2983	1-560	1-227	1-249
410-4	705	593	841	1-4422	1-8753	1-3041	1-496	1-914	1-227
423-8	721	607	842	1-4380	1-8753	1-3041	1-471	1-813	1-212
434-85	738	622	842	1-4407	1-8755	1-3030	1-442	1-722	1-199
444-25	747	639	845	1-4433	1-8807	1-3031	1-429	1-696	1-187
452-8	757	639	844	1-4465	1-8843	1-3027	1-419	1-677	1-182
460-4	765	643	841	1-4478	1-8854	1-3023	1-409	1-667	1-183
473-6	771	643	834	1-4489	1-8922	1-3060	1-396	1-654	1-184
484-95	775	652	841	1-4485	1-8964	1-3092	1-387	1-652	1-191
499-7	781	651	834	1-4476	1-9015	1-3135	1-368	1-652	1-208
519-7	792	649	820	1-4478	1-9052	1-3160	1-334	1-648	1-236
536-0	795	644	810	1-4509	1-9058	1-3104	1-329	1-650	1-242
544-5	797	640	803	1-4455	1-9010	1-3151	1-327	1-656	1-247
550-0	800	644	787	1-4475	1-8946	1-3089	1-360	1-738	1-278
555-0	800	635	794	1-4438	1-8758	1-2992	1-338	1-786	1-335
559-55	801	638	797
	1-658	2-362	1-276	Ratio of Highest to Lowest Ratio.	1-018	1-015	1-395	1-306	1-129

The percentage difference between the highest and lowest ratios in the various comparisons—including those with fluorbenzene—are given in the following Table :—

TABLE XX.

	$\frac{\text{CH}_3\text{OH}}{\text{C}_6\text{H}_5\text{F}}$	$\frac{\text{C}_2\text{H}_5\text{OH}}{\text{C}_6\text{H}_5\text{F}}$	$\frac{\text{C}_3\text{H}_7\text{OH}}{\text{C}_6\text{H}_5\text{F}}$	$\frac{\text{C}_2\text{H}_5\text{OH}}{\text{CH}_3\text{OH}}$	$\frac{\text{C}_3\text{H}_7\text{OH}}{\text{CH}_3\text{OH}}$	$\frac{\text{C}_3\text{H}_7\text{OH}}{\text{C}_2\text{H}_5\text{OH}}$
Absolute temperatures at corresponding pressures.....	10.5	13.8	14.6	3.0	4.0	1.5
Vapour-pressures at corresponding temperatures.....	391.0	602.0	1035.0	65.8	136.2	27.6
Molecular volumes of liquid at corresponding pressures	3.5	4.1	3.6	1.2	1.3	1.2
Molecular volumes of liquid at corresponding temperatures	3.2	3.0	3.4	0.9	1.8	1.5
Molecular volumes of saturated vapour at corresponding pressures	12.9	13.0	11.5	4.0	5.7	6.3
Molecular volumes of saturated vapour at corresponding temperatures	157.0	220.4	211.0	39.5	30.6	12.9

When the alcohols are compared with one another, the deviations of the ratios from constancy are much smaller than when fluorbenzene is taken as the standard substance, but they are still far outside the limits of experimental error. It may, perhaps, be said—as with the members of Group II. in the comparisons with fluorbenzene—that the generalizations offer a rough approximation to the truth.

Here, again, the comparison of the molecular volumes of saturated vapour at corresponding pressures is very much more satisfactory than at corresponding temperatures.

In the Philosophical Magazine for November 1890, p 417, it was pointed out by Prof. Orme Masson that the ratio of the molecular volumes (in the liquid state) of any two members of certain groups of nearly related carbon compounds, at their boiling-points under equal pressure, is equal to the ratio of those boiling-points, expressed on the absolute scale of temperature, or $\frac{V}{V'} = \frac{T}{T'}$.

In the same number of the Philosophical Magazine (p. 423)

I showed that Masson's relation is a special case of a more general one which should hold good if the generalizations of Van der Waals were strictly true. This relation may be expressed by the equations

$$\frac{v}{v'} = \frac{V}{V'} = \frac{T}{T'} \times \frac{p'}{p},$$

where v and v' are the molecular volumes of saturated vapour, V and V' those of liquid, and T and T' the boiling-points on the absolute scale of any two substances at corresponding pressures p and p' .

When the critical pressures are equal $p=p'$ and the equations become

$$\frac{v}{v'} = \frac{V}{V'} = \frac{T}{T'},$$

or the molecular volumes, whether of liquid or saturated vapour, at the boiling-points under equal pressure are proportional to the boiling-points expressed on the absolute scale.

That the relation cannot be strictly true when the critical pressures are different is obvious, since it is not generally true that the vapour-pressures at corresponding temperatures are corresponding pressures. It is still possible, however, that the relation may be true for the critical temperatures, pressures, and volumes. Unfortunately, the critical volumes have only been determined in a few cases, but it has been shown that the ratios of the molecular volumes at corresponding pressures do not vary within very wide limits; and it is, therefore, possible to test the relation in a limited form, taking the critical temperatures and pressures and the mean ratios of the molecular volumes at corresponding pressures.

A preliminary comparison was given in the paper referred to, but it was very incomplete, and there was a small error in some of the calculations owing to the misplacement of a figure in the critical pressure of fluorbenzene, which was given as 33190 instead of 33910.

As the experimental results have been considerably extended since that time and now include the molecular volumes of the saturated vapours, it is possible to give the Table in a complete form.

TABLE XXI.

Comparison of Values of $\frac{T}{T'} \times \frac{p'}{p}$ with Mean Ratios of

Molecular Volumes at corresponding Pressures.

Substances compared.	$\frac{T}{T'} \times \frac{p'}{p}$	Mean Ratios of Molecular Volumes.		Percentage difference between highest and lowest of the three values.
		Liquid.	Saturated Vapour.	
C_6H_5Cl/C_6H_5F	1.1308	1.1246	1.137	1.1
C_6H_5Br/C_6H_5F	1.1974	1.1802	1.189	1.5
C_6H_5I/C_6H_5F	1.2884	1.2772	1.282	0.9
C_6H_5Br/C_6H_5Cl	1.0589	1.0495	1.047	1.1
C_6H_5I/C_6H_5Cl	1.1394	1.1358	1.132	0.7
C_6H_5I/C_6H_5Br	1.0760	1.0822	1.080	0.6
C_6H_6/C_6H_5F9351	.9439	.946	1.2
CCl_4/C_6H_5F9861	1.0195	.993	3.4
$SnCl_4/C_6H_5F$	1.2771	1.2793	1.282	0.4
$(C_2H_5)_2O/C_6H_5F$	1.0468	1.0342	1.038	1.2
CH_3OH/C_6H_5F5203	.4228	.507	23.1
C_2H_5OH/C_6H_5F6537	.6153	.676	9.9
C_3H_7OH/C_6H_5F8533	.7995	.859	7.4
CH_3COOH/C_6H_5F8303	.6342	.593	40.0
C_2H_5OH/CH_3OH	1.2564	1.4534	1.329	15.7
C_3H_7OH/CH_3OH	1.6401	1.8890	1.693	15.2
C_3H_7OH/C_2H_5OH	1.3054	1.2999	1.274	2.5

For the halogen derivatives of benzene $p=p'$.

It will be seen that the differences between the three values are not very great, except in the comparisons of the three alcohols and acetic acid with fluorbenzene, and of ethyl and propyl alcohol with methyl alcohol. In the other cases, including the comparison of propyl with ethyl alcohol, the relation

$$\frac{v}{v'} = \frac{V}{V'} = \frac{T}{T'} \times \frac{p'}{p}$$

may be considered as approximately true with the limitations already stated.

The two substances—methyl alcohol and acetic acid, and to a small extent ethyl and propyl alcohol—are clearly exceptional in their behaviour, and it is of interest to note, in

comparing these substances with fluorbenzene, that while with acetic acid the ratios of the molecular volumes both of liquid and saturated vapour are very low, with methyl alcohol it is only in the liquid state that this is to be observed. This may again point to the existence of complex molecules of acetic acid in both the liquid and gaseous states, but of methyl alcohol in the liquid state only, except possibly very near the critical point.

With ethyl and propyl alcohols the ratios of the molecular volumes of liquid to those of fluorbenzene are also low, though not nearly to the same extent as with methyl alcohol.

Note on the Determination of Critical Constants.

Of the three critical constants, temperature, pressure, and volume, the first is the most easily determined, for by employing as heating-agents the vapours of pure liquids boiling under known pressures, the temperature is perfectly under control and is easily measured. Moreover, the presence of a very small amount of impurity does not influence the critical temperature of a substance to nearly the same extent as the critical pressure.

The critical pressure may also as a rule be determined without very much difficulty, provided that absence of impurity is ensured, but this point is of the utmost importance. In the case of substances that attack mercury at high temperatures, such as carbon tetrachloride and stannic chloride, the ordinary method of operation requires modification, and the calculations become more laborious, but otherwise the difficulty is not greatly increased.

The determination of the critical volume of a substance, even when perfectly pure and without action on mercury, is, however, a much more difficult matter. That this is so is evident from the form of the curves representing the relation of the volumes of liquid and of saturated vapour to the temperature or pressure, as shown in the accompanying diagrams (p. 264) constructed from the results with ethyl alcohol.

A very small alteration of temperature, such as 0.1° , at or just below the critical point, produces a considerable alteration in the volume ; therefore in order to obtain a direct reading of

the critical volume it is necessary that the substance shall be exactly at its critical temperature.

I have assumed that a substance is in this state when on rapidly increasing the volume somewhat above the critical volume the fall of temperature due to the expansion causes a momentary separation of liquid and vapour.

In order to determine the critical volume, I note the position of this temporary mark of division and then diminish the volume slightly. After waiting a few minutes for the temperature to become constant again, I increase the volume very slightly but rapidly, and again note the position of the mark

Fig. 1.

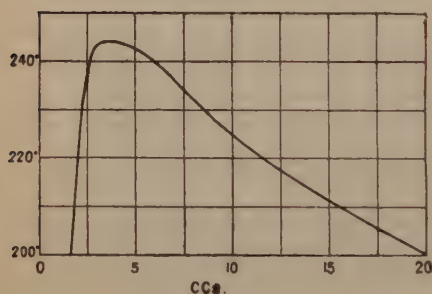
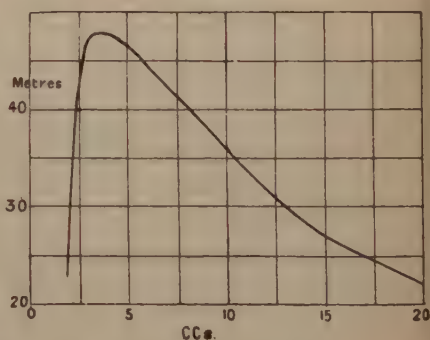


Fig. 2.



of division, which is now nearer to the top of the tube. Proceeding in this way it is possible under favourable conditions to make the substance occupy such a volume that a very slight but rapid expansion gives a temporary mark of division of liquid and vapour almost exactly at the top of the tube. This volume I take to be the critical volume, and I have succeeded in determining it directly in the case of benzene, fluorbenzene, and acetic acid; while with chlorobenzene, for which mercury vapour was employed as a jacket, it was only possible to obtain a rough approximation to the true volume on account of the slight unsteadiness of the temperature.

The critical volumes of the other substances may probably be ascertained with fair accuracy in the following manner:—

At low temperatures and pressures the ratios of the molecular volumes of liquid and saturated vapour of any one substance to those of fluorbenzene at corresponding temperatures and pressures differ somewhat widely as a rule, but as the critical point is approached the differences diminish and at the critical point itself all four values should, of course, be identical. It follows, therefore, that by mapping the ratios against temperature, four straight lines or curves should be obtained, which, when produced, should cut one another at the critical temperature, and the point of intersection should give the ratio of the molecular critical volume to that of fluorbenzene.

It will be seen from the accompanying diagrams, figs. 3 to 6, Plates VIII. & IX., that in the case of benzene and acetic acid the four curves do very nearly cut one another at the critical temperature, and the ratios of the critical volumes thus obtained agree very well with those calculated from the experimental results. The other substances will be considered in their order.

Halogen Derivatives of Benzene.—Accurate determinations of the volumes of liquid have been obtained only up to 280° , but the ratios are very nearly constant and are practically identical at corresponding temperatures and pressures; the extrapolation may therefore be considered justifiable. In the case of chlorobenzene a few approximate determinations have been made near the critical temperature; they give slightly higher ratios than those at lower temperatures, and agree rather more closely with the mean ratio of the absolute temperatures at corresponding (equal) pressures.

The molecular volumes of the saturated vapours give generally higher ratios than those of the liquid; this may be partly due to experimental error; but as the deviations are nearly all in the same direction, it seems hardly justifiable to attribute them entirely to this cause. It is evidently impossible to make use of these ratios in determining the critical volumes; but the mean ratios of the molecular volumes of liquid may probably be relied on to give fairly accurate results (Trans. Chem. Soc. lv. p. 517).

Carbon Tetrachloride.—As the observations extend to within

a few degrees of the critical temperature, there does not seem to be much room for error.

Stannic Chloride.—The molecular volumes of saturated vapour cannot be made use of, as the results at the highest temperatures are a little doubtful owing to slight decomposition of the substance; considerable extrapolation would also be necessary. The ratios for the liquid state give, however, very nearly straight lines, which may therefore probably be extrapolated without much error.

Ether.—The four curves would evidently be very close together at the critical temperature, but as the results with the saturated vapour are somewhat irregular it seems best to rely only upon those with the liquid.

Methyl Alcohol.—In this case the curves for the liquid state do not seem to agree well with those for the saturated vapour; I have taken the mean value, but it is evidently open to some doubt.

Ethyl and Propyl Alcohols.—The agreement is better with these alcohols, but the volumes of saturated vapour are somewhat irregular near the critical point.

Acetic Acid.—The four curves meet very satisfactorily at the critical temperature.

The mean ratios of the molecular volumes to those of fluor-benzene are given in the following Table, together with the molecular and specific critical volumes calculated from them,

TABLE XXII.

Substance.	Mean Ratio.	Molecular Volume. Calculated.	Specific Volume.		
			Calculated.	Observed.	Ramsay and Young.
C_6H_5F	2.43
C_6H_5Cl	1.1246	262	2.34	2.34-2.45
C_6H_5Br	1.1802	275	1.76
C_6H_5I	1.2772	298	1.47
C_6H_5944	220	2.83	2.82
CCl_4	1.010	235	1.53
$SnCl_4$	1.290	301	1.16
$(C_2H_5)_2O$	1.047	244	3.30	4.06
CH_3OH440	103	3.21	3.7
C_2H_5OH620	144	3.15	3.5
C_3H_7OH800	186	3.11	3.6
CH_3COOH630	147	2.45	2.46

taking the observed critical volume of fluorbenzene as correct. For the sake of comparison the specific volumes directly observed and also those previously adopted by Ramsay and myself are given.

It will be seen that the values for benzene and acetic acid, calculated from the mean ratios, agree very well with those directly observed, and this may be taken as evidence in favour of the accuracy of the method of direct measurement adopted. On the other hand, the calculated values for ether and the alcohols are considerably lower than those previously given by Ramsay and myself. But it may be pointed out that the critical temperature of ether was almost certainly slightly underestimated, owing to the employment of the vapour of methyl salicylate as a jacket. There is no reason to suspect any error in the determination of the critical pressure; and if the constants for Biot's formula be taken as correct, the calculated temperature corresponding to the critical pressure would be 194.4 instead of 193.8, and the higher value agrees well with a more recent determination by Ramsay, who in this case employed the vapour of quinoline as a jacket. The correction of the critical temperature of ether would give a lower value to the critical volume.

Again, Professor Tait has recently made an exhaustive mathematical investigation of the relations of pressure, temperature, and volume of several substances, including ether, and he has pointed out that the critical temperature is probably a little higher than 193.8, and that the critical volume of this substance is certainly lower than 4 cubic centim., and is more probably about 3.5 cubic centim., a value that agrees fairly well with that calculated by the method described.

In the case of the alcohols the critical volumes given by Ramsay and myself were estimated by inspection of the curves representing the relation of the specific gravities of liquid and saturated vapour to the temperature, and of the specific volumes to the pressures, and they were admittedly only approximate values.

M. Guye has shown (*Comptes Rendus*, cxii. p. 1257) that an approximate relation exists between the critical temperature, pressure, and volume of a substance and its theoretical

vapour-density compared with air. This is expressed by the equations

$$d = 1146 \frac{\delta\theta}{\pi(1070 + \theta)} = \frac{M}{28.87};$$

where δ is the specific gravity at the critical point compared with water at 4° , θ the absolute critical temperature, π the critical pressure, and M the molecular weight. The critical volumes of ether and the alcohols previously given by Ramsay and myself are certainly in better agreement with this relation than those which I now find; but it may be mentioned that, with the exception of iodobenzene, all the substances referred to in this paper give somewhat higher values of d than the quotient $\frac{M}{28.87}$ if the molecular volumes in

Table XXII. be taken as correct.

It may, I think, be safely stated that the true critical volumes are not lower than those directly observed, but it is possible that they may be a little higher. It is therefore perhaps too much to say that the values given in the table are definitely established, but I am inclined to think that they may be accepted as fairly close approximations to the truth.

If that is so, it would appear from Table XVII. that *the ratio of the actual critical density to the theoretical density is for very many substances about 4.4*. Ethyl and propyl alcohol give somewhat higher values, while for methyl alcohol and acetic acid, especially the latter, the ratios are extremely high.

Prof. Ramsay said the results proved that Van der Waals's generalizations were only rough approximations, and he suggested that some force had been neglected, or a term omitted from the equations. Perhaps the assumption that the molecules are incompressible was not correct. He also strongly protested against the tacit assumption of the Van der Waals's laws, and deductions made therefrom, which had recently become so common, particularly in German text-books.

Prof. Perry inquired whether the quantities a , b , and α had been determined for different substances and found to be constant.

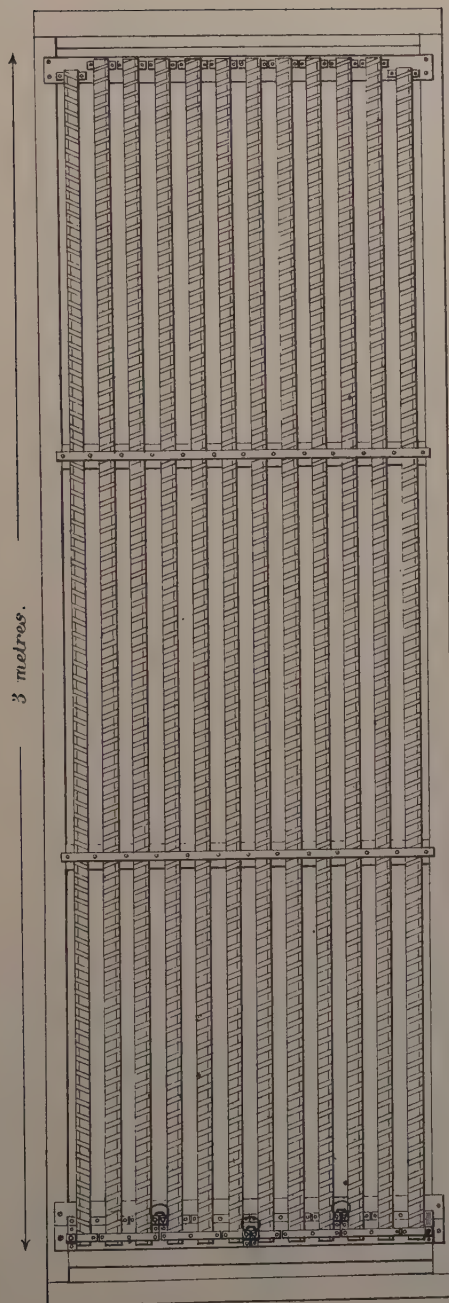


Fig. 1

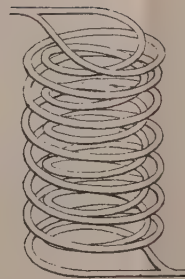


Fig. 2.
*Right and Left Handed
Spirals in Parallel.
Opposing one another.*

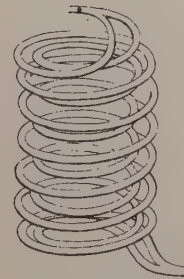


Fig. 4.
*Two Right Handed Spirals
in Parallel, Assisting one
another.*

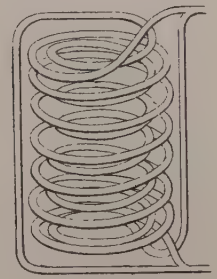


Fig. 5.
*Two Right Handed Spirals
in Parallel, Opposing one
another.*

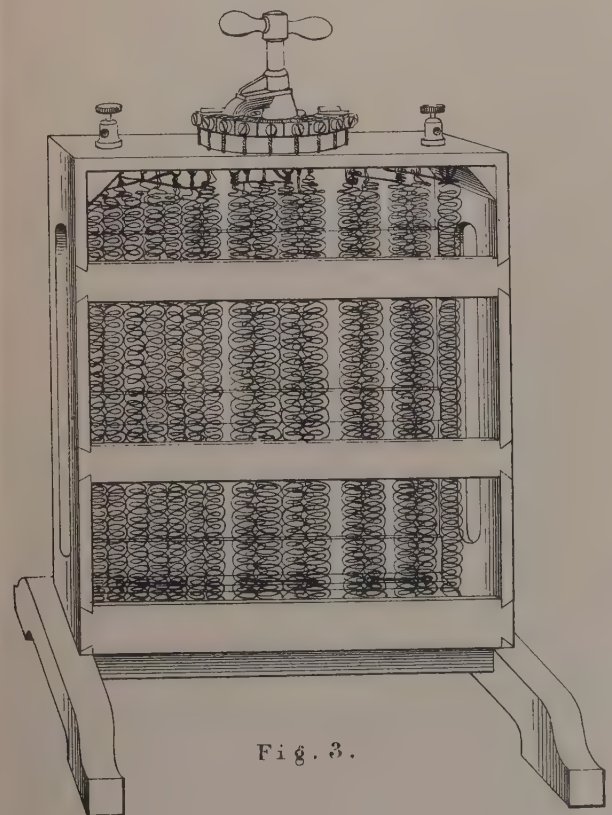
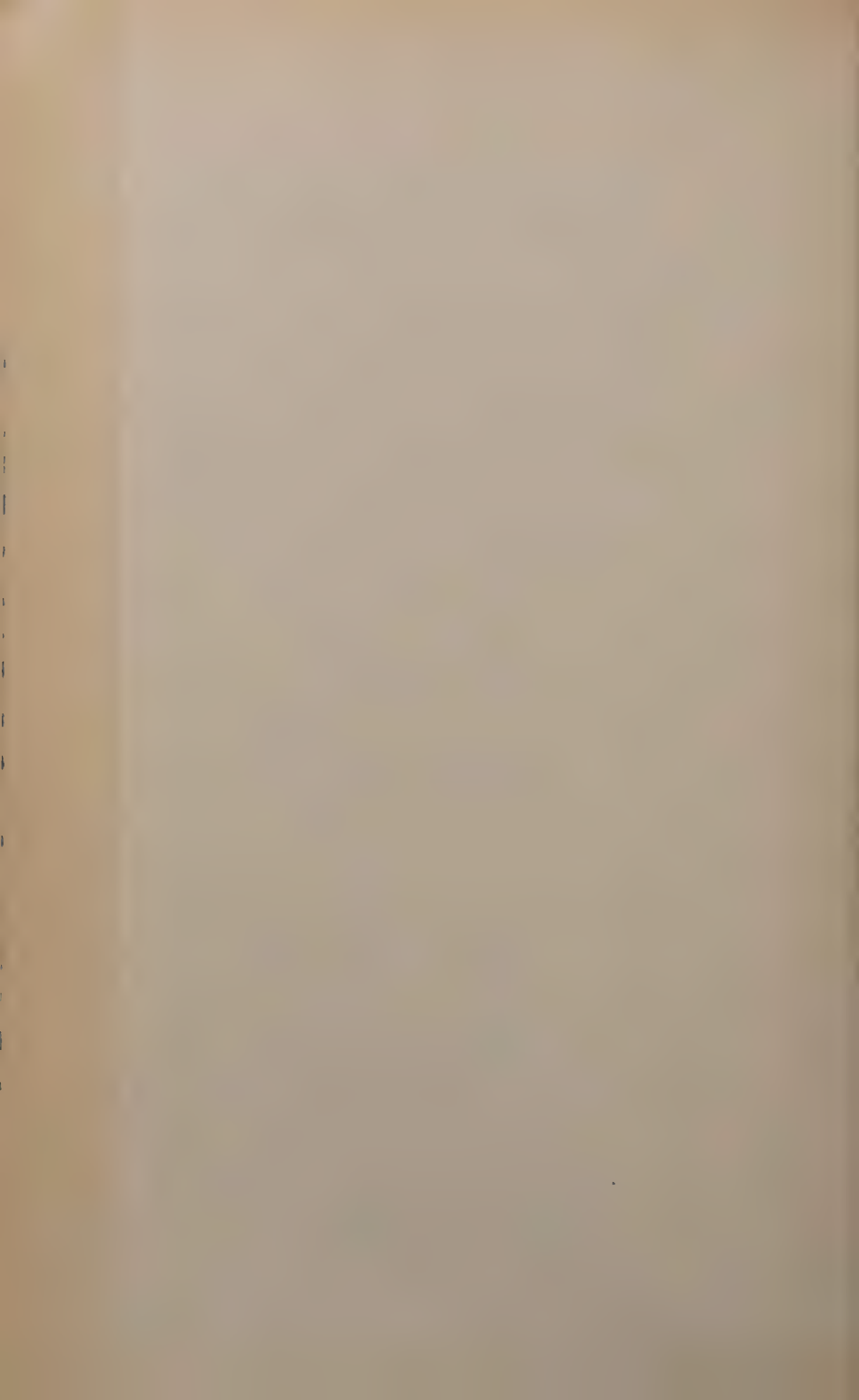


Fig. 3.



Prof. Ramsay said that for substances in states analogous to those of perfect gases the quantities were approximately constant, but when the liquid state was approached this was no longer true. According to Prof. Tait the two states were not continuous.

Prof. Herschel remarked that Prof. Tait had established his law on the assumption that the co-volume is four times the volume occupied by the molecules. This law, he said, had been amply verified by experiments on explosions.

Dr. Burton, referring to Prof. Ramsay's remarks on the compressibility of molecules, said the law of force between attracting molecules should be accurately known before any deductions were made; and he pointed out that at constant volume the pressure should be proportional to the absolute temperature if allowance be made for the negative pressure of attraction.

Mr. Blakesley, in speaking of molecular forces, said he had observed that when water is allowed to evaporate from glass, a furrow is formed in the glass which marks out the original boundary of the liquid. To all appearances the particles of glass are torn away by the molecular forces acting along the boundary.

XXIX. *The Construction of Non-Inductive Resistances.*

By Prof. W. E. AYRTON, *F.R.S.*, and T. MATHER*.

[Plate X.]

WITH all electric methods devised for measuring the power given by a varying current to a circuit that may possess inductance or capacity it is necessary to employ a resistance which shall have zero inductance, consequently it is important to consider how such resistances may most easily be made.

If the inductance of this nominally non-inductive resistance be not zero, the error thus introduced into the measurement can still be made relatively unimportant if the time-constant of this portion of the circuit be made small. For, as shown

* Read June 26, 1891.

by Dr. Sumpner and one of us, in a paper* read before this Society on June 12th of this year, the following proportion holds true in all the nine methods of measuring power there considered:—

$$\frac{\text{the watts as measured}}{\text{the true watts}} = \frac{1 + \tan \theta \cdot \tan \phi}{1 + \tan^2 \phi},$$

where θ and ϕ are the angles of phase-difference between the current and the P.D. for the circuit the power given to which we desire to measure, and for the auxiliary circuit respectively.

Now for any given configuration of a circuit, the time-constant will be the smaller the higher the specific resistance of the conductive material; hence it is very desirable to use a material of high specific resistance, like carbon. For this reason, glow-lamps constitute valuable small time-constant resistances, but carbon has the disadvantage that its resistance varies rapidly with temperature. Hence, since with the methods of measuring power referred to it is necessary to know the resistance of the non-inductive circuit at the moment of making the measurement, it follows that if carbon be employed an extra measurement has to be made.

When the non-inductive resistance is put in series with the circuit the power given to which we desire to measure, as, for example, with the three-voltmeter method of measuring power, a measurement of the resistance of the glow-lamps merely means the reading of an extra instrument at the moment the power-test is made; but when the two circuits are joined in parallel, as, for example, with the one-voltmeter and two-ammeter method of measuring power, a measurement of the resistance of the glow-lamps cannot be made simultaneously with the power-test, since to do this would require the introduction of an ammeter, and therefore of inductance, into the circuit, which should be non-inductive.

Further, carbon is unsuitable for portable resistances on account of its brittle nature. Platinoid, on the other hand, is flexible, has a low temperature-coefficient, and a high specific resistance, though not of course nearly as high as

* "Alternate Current and Potential Difference Analogies in the Methods of Measuring Power," *ante*, p. 172.

that possessed by carbon. Platinoid therefore appeared to us to be the best material to employ in the construction of non-inductive resistances to be used for power-tests.

It is well known that a wire doubled on itself has a very small inductance, which approximates to 3·77 times the total length of the wire in centimetres as the parts approach each other. This value can, as Maxwell pointed out, be reduced by using flat strips instead of round wires; and, if the strips be bare and be placed vertically, it is clear that the cooling action will be considerable, so that relatively strong currents will produce but little rise in temperature or increase of resistance.

We therefore decided in 1887 to construct the non-inductive resistance seen in fig. 1, Plate X., consisting of twelve platinoid strips, each 6 metres long, 4 centim. wide, and 0·25 millim. thick. Each strip is doubled on itself, two layers of carefully shell-lacqued silk, 0·075 millim. thick, being inserted between the front and back portions. The whole is bound together by means of a narrow silk ribbon wrapped round spirally, considerable gaps being left between the spires of the silk ribbon so that the platinoid should have plenty of free surface for cooling.

For the purpose of expelling moisture from the silk and the shell-lac varnish, when the resistance-strips are first put up, a current was passed through each strip strong enough to make it fairly hot. During this heating frequent short-circuitings occurred from rough points of the metal piercing the silk, a single layer only of which was originally employed to separate the front and back portions. An additional layer of silk was therefore inserted. Trouble was also experienced from the rough edges of the metal causing short-circuits; but by cutting the silk wider than the metal and by folding the edge of the silk over the edge of the platinoid strip, this difficulty was overcome.

The doubled strips are permanently joined up in sets of three, and to the ends of each set are soldered mercury-cups and binding-screws. The four sets of three can be joined up in series, or in parallel, or in parallel-series by bridge-pieces dipping into the mercury-cups, and, when all are in series, the resistance of the 72 metres of strip at 15° C. is 2·932

ohms. The resistance of the arrangement does not alter by more than one tenth per cent. when a current of 15 amperes is passing through each strip.

The wooden frame which carries this series of platinoid strips hangs on paraffined ebonite pegs attached to one of the laboratory walls, so that it is well insulated.

The inductance of the set of strips, even when all are in series, is so small that we have not been able to measure it even with the secohmmeter. Some of the tests have given indications of a small negative result, which, if true, would mean that the capacity-effect slightly overbalanced the inductance; but a calculation, which we have made, appears to show that with the dimensions in question such a result is impossible. We can therefore only conclude that this resistance-frame, which was constructed by two of our former students, Messrs. C. G. Lamb and E. W. Smith, fulfils the object for which it was intended so well that the inductance cannot be detected with certainty by any test that we have hitherto tried. We therefore have decided to duplicate the arrangement, the wooden framework (only one half of which is seen in fig. 1) having been constructed large enough to hold a second set of strips.

Another method of constructing non-inductive resistances for large currents, which has been in use at the Central Institution for the past eighteen months, is illustrated in fig. 2, and consists in winding two bare platinoid wires of equal length and thickness into two spirals, one right-handed, the other left-handed, the diameters of the two spirals differing slightly from each other so that one spiral can be placed inside the other. They are then connected up in parallel, so that when a current is sent through them it circulates clockwise round one spiral and counterclockwise round the other, the magnetic effects of the two thus tending to neutralize one another and to produce a combination with small inductance.

The inductance of coils constructed in this way is almost as small as if the wires were doubly wound like an ordinary resistance-coil, *but they possess the great advantage that parts differing much in potential are not close to one another.* The wires therefore require no insulating covering, for no

harm will occur if one spiral accidentally touches the other, provided that reasonable care has been taken to space the convolutions fairly uniformly. Further, as the cooling surface, for a given total cross section of the conductor, is much greater for two concentric spirals than for a single spiral of thicker wire, it follows that this method of constructing a resistance not only greatly reduces the inductance, but also enables a much larger current to be carried for a given variation of resistance produced by the current. Hence for a given current and for a given permissible rise of temperature finer wire can be used; the coils can therefore be made shorter, and the inductance for this reason still further lessened.

Lastly, even when such right- and left-handed spirals are traversed by a direct current, much less disturbance is produced in a neighbouring delicate galvanometer than if a singly-wound spiral resistance be employed; so that these right- and left-handed spirals of bare wire are valuable in the construction of resistance-frames for large direct as well as for large alternating currents.

A number of portable resistance-coils have been constructed in this way for general use in the laboratories of the Central Institution. One of these is seen in fig. 3. It has also been found convenient to fit up a number of stationary right- and left-handed concentric spirals of platinoid in the dynamo laboratory, for use in the regular experiments with alternate currents.

Table I. gives the particulars of some portable sets of stand coils in the laboratories of the Central Institution, the second three of which have been composed of right- and left-handed spirals in the way just described and illustrated in fig. 3.

Table II. gives the results of experiments made on the sets marked 2, 3, 4, 5, 6 by means of the secohmmeter; the set marked 6 being tested when each pair of coils were joined up in parallel so as to increase the inductance as well as when joined up so as to diminish it.

TABLE I.
Particulars of Portable Stand Coils.

Mark on the Stand.	Resistance, in Ohms.	Number of Spirals in the Set.	Length of each Spiral.	Number of Convolutions in each Spiral.		Outside diameter of the Convolutions.		Size of the Wire (S.W.G.)
1	1.8	6	inches. 18 $\frac{3}{4}$	78		inches. 1 $\frac{1}{2}$		No. 12
2	2.54	12	16 $\frac{1}{2}$	63		1 $\frac{3}{8}$		No. 10
3	8.59	12	16 $\frac{1}{2}$	84		1 $\frac{1}{4}$		No. 13
				Outer.	Inner.	Outer.	Inner.	
4	1.4	20 double, that is 40 coils in all. " " " "	16 $\frac{1}{2}$	15	21	inches. 1 $\frac{1}{8}$	inches. $\frac{3}{8}$	No. 13
5	5.0		16 $\frac{1}{2}$	27	37	1 $\frac{1}{8}$	$\frac{1}{4}$	No. 15 $\frac{1}{2}$
6	99		17	54	79	$\frac{3}{4}$	$\frac{1}{2}$	No. 27

TABLE II.
Portable Platinoid Spiral Resistances (fig. 3).

Mark on the Stand.	Winding.	Resistance, in Ohms.	Inductance, in Secohms.	Time-constant, in Seconds.
2 ...	Set of Singly-Wound Spirals ...	2.54	0.000059	23×10^{-6}
3 ...	Set of Single-Wound Spirals ...	8.59	0.00015	18×10^{-6}
6 {	Set of Doubly-Wound Spirals; Current circulating in all in the Same Direction	98.1	0.0021	21×10^{-6}
6 {	Same set of Doubly-Wound Spirals; Current circulating in Opposite Directions.....	99	0.00013	1.3×10^{-6}
4 {	Set of Doubly-Wound Spirals; Current circulating in Opposite Directions	1.4	0.000004	2.8×10^{-6}
5 {	Set of Doubly-Wound Spirals; Current circulating in Opposite Directions	5.0	0.0000094	1.9×10^{-6}

The mean time-constant, therefore, of one of our sets of spirals, whether singly-wound or doubly-wound, provided that the current circulates in all the convolutions in the same

direction, is about 21×10^{-6} ; whereas if the spirals be doubly-wound and the current be circulating in opposite directions in the two sets of spirals, the mean time-constant is about 2×10^{-6} , or about one tenth of the preceding.

As there happens to be in the laboratory a solenoid, the silk-covered copper wire on which is wound in a series of concentric sections, it was thought interesting to compare the time-constant of two of the sections joined up in parallel, when the current circulated round both in the same direction, with the time-constant when it circulated in opposite directions. Figs. 4 and 5, Plate X., show the direction of the current in the two cases.

	Resistance, in Ohms.	Self-induction, in Sechoms.	Time-constant, in Seconds.
Currents flowing round both coils } in the same direction (fig. 4) ... }	0.346	0.00059	17×10^{-6}
Currents flowing round the coils } in the opposite direction (fig. 5) }	0.346	0.000024	0.69×10^{-6}

The self-induction and time-constant have therefore been reduced to one twenty-fourth part by sending the currents in opposite directions round the two, but the method of winding shown in fig. 5 could not, of course, be employed with the non-inductive resistances constructed of bare wire for large currents described in this communication; since the fundamental condition—that *parts differing much in potential should not be near one another*—would not be fulfilled.

XXX. A Theory concerning the Constitution of Matter.

By CHARLES V. BURTON, D.Sc.*

THE theory described in the following pages is based essentially on one fundamental hypothesis; without the aid of any further assumptions, equations of motion can be deduced, and these, when simplified by certain conditions of symmetry,

* Read November 20, 1891.

lead at once to Newton's Laws. In dealing—vaguely enough—with other problems, such as gravitation and the discrete nature of atoms, it is found necessary to make further limitations; and a few suggestions are also made whose nature is purely speculative. But there remains the one central idea, whose development is the especial object of this paper, and in the concluding paragraphs the doctrine will be stated in its most general form and the arguments once more briefly enumerated.

1. Space, so far as we know it, is filled with a medium, whose ultimate nature may be fluid, but which, owing to turbulent motion or some other cause, has elastic properties resembling in some respects those of a solid. The resistance offered by such a medium to the motion of material bodies presents a problem of some difficulty, *so long as we suppose an atom to consist always of the same portion of æthereal or other substance*; but further on it will be shown that, with a different assumption, the question may assume a new aspect. Before leaving this subject, however, I may point out what appears to me a very serious difficulty of the present view: a perfect vacuum is (at least very nearly) a perfect insulator, and air also insulates well, so that the æther surrounding an electrically charged terrestrial body may remain in a state of stress for a considerable period without appreciable progressive yielding, while at the same time this charged body is being whirled through the æther, which continually gives way like an almost perfect liquid.

2. *Fundamental Assumption. Strain-Figure.*—Consider a region, either infinite or having very distant boundaries, and filled with a homogeneous isotropic elastic medium, whose condition throughout is one of stable equilibrium for small strains of any type. Let the medium now be strained, and held in its strained condition by some compelling agency: there will be a corresponding distribution of stress in the medium, and, provided the strain has at no point too great a value, the original condition will be completely regained after the compelling agency has been removed. But suppose that, instead, the medium is strained further and further from its initial state, and suppose that the restoring stresses do not

always increase with the strain, but that beyond a certain point in the process they begin to fall off in value, until at last a point is reached at which the general tendency of the stress is to further increase the strain. If the compelling agency is now withdrawn, the medium will subside into a new condition of stable equilibrium, involving stress and strain at every point. The state of things thus impressed on the medium is, according to my view, an atom or a constituent of an atom; it will hereafter be referred to as a "*strain-figure*," and we may now proceed to examine its dynamical properties.

3. *Rigid Body Displacements*.—A strain-figure, being of itself in equilibrium, will remain in equilibrium if transferred to some other portion of the medium, or if its orientation with respect to the medium is changed (for originally the medium was homogeneous and isotropic); we may therefore give to the strain-figure any displacement of which a rigid body would be capable, and the resulting condition of the medium will be one of equilibrium; there is no statical resistance to such displacement, and no question of the medium giving way.

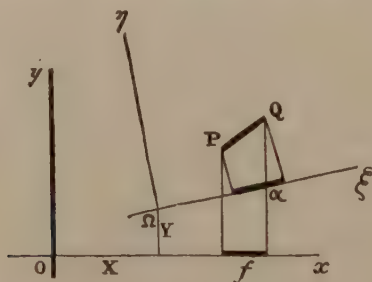
4. *Equations of Motion*.—If a strain-figure is in motion through the medium, certain conditions must be satisfied in order that its degrees of freedom may not be more than those of a rigid body; in order, that is, that the strain-figure may retain the same form as if it were at rest in the medium. For suppose that disturbances of the same type as the strains in the strain-figure are propagated through the medium with velocity V ; then obviously a necessary condition is that the translational velocity of the strain-figure must be very small compared with V , the rotational motion being (in general) subject to a similar restriction, which cannot be quite so simply expressed. If V is (as I imagine) the velocity with which gravitation is propagated, it is a quantity whose finiteness has not yet been demonstrated, and compared with which all known molar and molecular motions are extremely slow. Another condition to be fulfilled by the motion of the strain-figure is this: the acceleration must be very small compared with V/T , where T is the radius of the smallest spherical surface within which the whole appreciable *effective mass* (17 a) of the strain-figure may be considered to reside.

Assuming, then, that the necessary conditions are satisfied,

and that the strain-figure has only six degrees of freedom, let us choose three rectangular axes (those of ξ, η, ζ , with origin Ω) fixed in the strain-figure, and three other rectangular axes (those of x, y, z , with origin O) fixed in the medium and in space; let the coordinates of Ω referred to Ox, Oy, Oz be (X, Y, Z) , and let the direction-cosines of the ξ, η, ζ axes be given by the scheme

	$x-X$	$y-Y$	$z-Z$	
ξ	l_1	m_1	n_1	
η	l_2	m_2	n_2	$\dots \dots (1)$
ζ	l_3	m_3	n_3	

(For convenience the diagram is only drawn in two dimensions.)



Let $P(\xi, \eta, \zeta)$ be the *undisturbed* position of a certain volume-element of æther (that is, the position which this element would occupy if the strain-figure were non-existent), and let $Q(\xi + \alpha, \eta + \beta, \zeta + \gamma)$ be the *actual* position of the same element. Then, in accordance with the geometrical conditions of motion,

$$\alpha, \beta, \gamma \text{ are functions of } \xi, \eta, \zeta \text{ only; } \dots \dots (2)$$

they are the components of displacement of the element whose undisturbed position would have been ξ, η, ζ .

If the components of PQ parallel to Ox, Oy, Oz be called f, g, h respectively, we shall have, after the manner of (1), the scheme

	f	g	h	
α	l_1	m_1	n_1	
β	l_2	m_2	n_2	$\dots \dots \dots (3)$
γ	l_3	m_3	n_3	

Let the *original* density of the æther be taken as unity; and consider that portion of æther which, in the *undisturbed* condition of the medium, would have filled the volume-element $d\xi d\eta d\zeta$ with coordinates (ξ, η, ζ) . Since f, g, h are the displacements parallel to Ox, Oy, Oz , of the portion of æther so defined, $\dot{f}, \dot{g}, \dot{h}$ will be the actual velocity-components of this portion, and the energy due to the motion of the strain-figure is

$$T = \frac{1}{2} \iiint (\dot{f}^2 + \dot{g}^2 + \dot{h}^2) d\xi d\eta d\zeta. \quad . \quad . \quad . \quad (4)$$

Now let the axes $\Omega\xi, \Omega\eta, \Omega\zeta$ be instantaneously coincident in direction with Ox, Oy, Oz respectively, so that

$$l_1 = m_2 = n_3 = 1; \quad l_2 = l_3 = m_3 = m_1 = n_1 = n_2 = 0, \quad . \quad . \quad (5)$$

and

$$\dot{l}_1 = \dot{m}_2 = \dot{n}_3 = 0; \quad \dot{n}_2 = -\dot{m}_3 = \omega_1; \quad \dot{l}_3 = -\dot{n}_1 = \omega_2; \quad \dot{m}_1 = -\dot{l}_2 = \omega_3, \quad (6)$$

where $\omega_1, \omega_2, \omega_3$ are the angular velocity-components of the strain-figure about the axes of ξ, η, ζ or of x, y, z ; relations which, in general, are only instantaneously true.

From (3),

$$\dot{f} = l_1 \dot{\alpha} + l_2 \dot{\beta} + l_3 \dot{\gamma} + \dot{l}_1 \alpha + \dot{l}_2 \beta + \dot{l}_3 \gamma;$$

and writing

$$\frac{\partial \alpha}{\partial \xi} \equiv \alpha_1; \quad \frac{\partial \alpha}{\partial \eta} \equiv \alpha_2; \quad \frac{\partial \alpha}{\partial \zeta} \equiv \alpha_3; \quad \&c., \quad . \quad . \quad (7)$$

the last equation becomes

$$\begin{aligned} \dot{f} = l_1(\alpha_1 \dot{\xi} + \alpha_2 \dot{\eta} + \alpha_3 \dot{\zeta}) + l_2(\beta_1 \dot{\xi} + \beta_2 \dot{\eta} + \beta_3 \dot{\zeta}) + l_3(\gamma_1 \dot{\xi} + \gamma_2 \dot{\eta} + \gamma_3 \dot{\zeta}) \\ + \dot{l}_1 \alpha + \dot{l}_2 \beta + \dot{l}_3 \gamma. \quad . \quad . \quad . \quad (8) \end{aligned}$$

Again, from (1),

$$\xi = l_1(x - X) + m_1(y - Y) + n_1(z - Z),$$

so that

$$\begin{aligned} \dot{\xi} = \dot{l}_1(l_1 \dot{\xi} + l_2 \dot{\eta} + l_3 \dot{\zeta}) + \dot{m}_1(m_1 \dot{\xi} + m_2 \dot{\eta} + m_3 \dot{\zeta}) + \dot{n}_1(n_1 \dot{\xi} + n_2 \dot{\eta} + n_3 \dot{\zeta}) \\ - l_1 \dot{X} - m_1 \dot{Y} - n_1 \dot{Z}; \end{aligned}$$

or, using (5) and (6),

$$\left. \begin{aligned} \dot{\xi} &= \omega_3 \dot{\eta} - \omega_2 \dot{\zeta} - \dot{X}. \\ \dot{\eta} &= \omega_1 \dot{\zeta} - \omega_3 \dot{\xi} - \dot{Y}, \\ \dot{\zeta} &= \omega_2 \dot{\xi} - \omega_1 \dot{\eta} - \dot{Z} \end{aligned} \right\}$$

Similarly

and

Substituting these values in (8), and again using (5) and (6), we obtain

$$\left. \begin{aligned}
 \dot{f} &= -\alpha_1 \dot{X} - \alpha_2 \dot{Y} - \alpha_3 \dot{Z} + \omega_1(\alpha_2 \zeta - \alpha_3 \eta) + \omega_2(\alpha_3 \xi - \alpha_1 \zeta + \gamma) \\
 &\quad + \omega_3(\alpha_1 \eta - \alpha_2 \xi - \beta). \\
 \text{Similarly} \\
 \dot{g} &= -\beta_1 \dot{X} - \beta_2 \dot{Y} - \beta_3 \dot{Z} + \omega_1(\beta_2 \zeta - \beta_3 \eta - \gamma) + \omega_2(\beta_3 \xi - \beta_1 \zeta) \\
 &\quad + \omega_3(\beta_1 \eta - \beta_2 \xi + \alpha), \\
 \text{and} \\
 \dot{h} &= -\gamma_1 \dot{X} - \gamma_2 \dot{Y} - \gamma_3 \dot{Z} + \omega_1(\gamma_2 \zeta - \gamma_3 \eta + \beta) + \omega_2(\gamma_3 \xi - \gamma_1 \zeta - \alpha) \\
 &\quad + \omega_3(\gamma_1 \eta - \gamma_2 \xi).
 \end{aligned} \right\} \quad (9)$$

If these values are substituted in (4), we obtain immediately a complete expression for the energy due to the motion of the strain-figure; and this expression involves only \dot{X} , \dot{Y} , \dot{Z} , ω_1 , ω_2 , ω_3 , and quantities which remain constant throughout the motion. We may therefore apply the principle of moving axes to find the components F, G, H of effective force on the strain-figure; thus

$$F = \frac{d}{dt} \frac{\partial T}{\partial \dot{X}} - \omega_3 \frac{\partial T}{\partial \dot{Y}} + \omega_2 \frac{\partial T}{\partial \dot{Z}},$$

which by means of (4) and (9) becomes

$$\begin{aligned}
 F &= (11)\ddot{X} + (12)\ddot{Y} + (31)\ddot{Z} \\
 &\quad - (12)\omega_3 \dot{X} - (22)\omega_3 \dot{Y} - (23)\omega_3 \dot{Z} \\
 &\quad + (31)\omega_2 \dot{X} + (23)\omega_2 \dot{Y} + (33)\omega_2 \dot{Z} \\
 &\quad + \{(31\eta) - (12\zeta) + (\beta_1\gamma)\}\dot{\omega}_1 + \{(11\zeta) - (31\xi) + (\gamma_1\alpha)\}\dot{\omega}_2 \\
 &\quad + \{(12\xi) - (11\eta) + (\alpha_1\beta)\}\dot{\omega}_3 \\
 &\quad + \{(31\zeta) - (33\xi) + (\gamma_3\alpha)\}\omega_2^2 + \{(12\eta) - (22\xi) - (\alpha_2\beta)\}\omega_3^2 \\
 &\quad + \{2(23\xi) - (12\zeta) - (31\eta) + (\alpha_3\beta) - (\gamma_2\alpha)\}\omega_2\omega_3 \\
 &\quad + \{(22\zeta) - (23\eta) + (\beta_2\gamma)\}\omega_3\omega_1 + \{(33\eta) - (23\zeta) + (\beta_3\gamma)\}\omega_1\omega_2, \quad (10)
 \end{aligned}$$

with similar values for G and H, where we write

$$\left. \begin{aligned}
 \iiint (\alpha_1^2 + \beta_1^2 + \gamma_1^2) d\xi d\eta d\zeta &\equiv (11), \text{ \&c.,} \\
 \iiint (\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3) d\xi d\eta d\zeta &\equiv (23) = (32), \text{ \&c.,} \\
 \iiint (\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3)\xi d\xi d\eta d\zeta &\equiv (23\xi) \equiv (32\xi), \text{ \&c.,} \\
 \iiint (\beta_1\gamma - \gamma_1\beta) d\xi d\eta d\zeta &\equiv (\beta_1\gamma) \equiv -(\gamma_1\beta), \text{ \&c.}
 \end{aligned} \right\} \quad (11)$$

Again, if L, M, N are the effective couples,

$$\begin{aligned}
 L = & \frac{d}{dt} \cdot \frac{\partial T}{\partial \omega_1} - \dot{Z} \frac{\partial T}{\partial \dot{Y}} + \dot{Y} \frac{\partial T}{\partial \dot{X}} - \omega_3 \frac{\partial T}{\partial \omega_2} + \omega_2 \frac{\partial T}{\partial \omega_3} \\
 = & \{ (12\zeta) - (31\eta) - (\beta 1\gamma) \} \ddot{X} + \{ (23\eta) - (22\zeta) + (\beta 2\gamma) \} \ddot{Y} \\
 & + \{ (33\eta) - (23\zeta) + (\beta 3\gamma) \} \ddot{Z} \\
 & + (23) \dot{Y}^2 - (23) \dot{Z}^2 + \{ (33) - (22) \} \dot{Y} \dot{Z} - (12) \dot{Z} \dot{X} + (31) \dot{X} \dot{Y} \\
 & + \{ (12\xi) - (11\eta) + (\alpha 1\beta) \} \omega_2 \dot{X} + \{ (31\xi) - (11\zeta) - (\gamma 1\alpha) \} \omega_3 \dot{X} \\
 & + \{ (33\eta) - (23\zeta) + (\beta 3\gamma) \} \omega_1 \dot{Y} + \{ (22\zeta) - (23\eta) - (\beta 2\gamma) \} \omega_1 \dot{Z} \\
 & + \{ (22\xi) - (33\xi) + (31\zeta) - (12\eta) + (\gamma 3\alpha) + (\alpha 2\beta) \} (\omega_2 \dot{Y} - \omega_3 \dot{Z}) \\
 & + \{ 2(23\xi) - (12\zeta) - (31\eta) + (\alpha 3\beta) - (\gamma 2\alpha) \} (\omega_3 \dot{Y} + \omega_2 \dot{Z}) \\
 & + \{ (22\zeta^2) + (33\eta^2) - 2(23\eta\zeta) - (\beta 2\gamma\zeta) + (\beta 3\gamma\eta) \} \dot{\omega}_1 \\
 & + \{ (23\zeta\xi) + (31\eta\zeta) - (33\xi\eta) - (12\zeta^2) - (\gamma 2\alpha\zeta) + (\gamma 3\alpha\eta) \\
 & \quad - (\beta 3\gamma\xi) + (\beta 1\gamma\zeta) \} \dot{\omega}_2 \\
 & + \{ (12\eta\zeta) + (23\xi\eta) - (22\zeta\xi) - (31\eta^2) - (\alpha 2\beta\zeta) + (\alpha 3\beta\eta) \\
 & \quad - (\beta 1\gamma\eta) + (\beta 2\gamma\xi) \} \dot{\omega}_3 \\
 & + \{ (\alpha 3\beta\xi) - (\alpha 1\beta\zeta) + (\gamma 1\alpha\eta) - (\gamma 2\alpha\xi) + (\beta\gamma) \} (\omega_3^2 - \omega_2^2) \\
 & + \{ \alpha 2\beta\xi - (\alpha 1\beta\eta) + (\gamma 3\alpha\xi) - (\gamma 1\alpha\zeta) + (\beta^2) - (\gamma^2) \} \omega_2 \omega_3 \\
 & + \{ (33\xi\eta) + (12\zeta^2) - (23\zeta\xi) - (31\eta\zeta) + (\gamma 2\alpha\zeta) - (\gamma 3\alpha\eta) \\
 & \quad + (\beta 3\gamma\xi) - (\beta 1\gamma\zeta) \} \omega_3 \omega_1 \\
 & + \{ (12\eta\zeta) + (23\xi\eta) - (22\zeta\xi) - (31\eta^2) - (\alpha 2\beta\zeta) + (\alpha 3\beta\eta) \\
 & \quad - (\beta 1\gamma\eta) + (\beta 2\gamma\xi) \} \omega_1 \omega_2, \quad \dots \quad (12)
 \end{aligned}$$

with similar values for M and N, where, in addition to (11), we have

$$\left. \begin{aligned}
 \iiint (\alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2) \eta \zeta \, d\xi \, d\eta \, d\zeta &\equiv (12\eta\zeta) \text{ \&c.}, \\
 \iiint (\beta_1 \gamma - \gamma_1 \beta) \zeta \, d\xi \, d\eta \, d\zeta &\equiv (\beta 1\gamma\zeta) \text{ \&c.}, \\
 \iiint \beta \gamma \, d\xi \, d\eta \, d\zeta &\equiv (\beta\gamma) \text{ \&c.}
 \end{aligned} \right\} \quad (13)$$

Mr. G. H. Bryan has pointed out to me that the equations of motion (10) and (12) are of exactly the same form as those of a solid immersed in a perfect liquid.

5. If U is the force-function of the impressed forces, we shall have of course

$$F = \frac{\partial U}{\partial X}, \quad G = \frac{\partial U}{\partial Y}, \quad H = \frac{\partial U}{\partial Z},$$

and evidently

$$\frac{\partial U}{\partial X} = \iiint \left(a \frac{\partial f}{\partial X} + b \frac{\partial g}{\partial X} + c \frac{\partial h}{\partial X} \right) d\xi' d\eta' d\zeta';$$

where $d\xi' d\eta' d\zeta'$ is a volume-element in the *actual* condition of the medium, f, g, h the displacements of the æther which now fills this volume-element, and $a d\xi' d\eta' d\zeta'$ &c., the forces exerted on the element of æther. Remembering (1), (3), (5), and (7), we may write the last equation

$$\frac{\partial U}{\partial X} = -\iiint (a\alpha_1 + b\beta_1 + c\gamma_1) d\xi' d\eta' d\zeta'; \quad . \quad . \quad (14)$$

$\frac{\partial U}{\partial Y}$ and $\frac{\partial U}{\partial Z}$ having similar values, and $\frac{\partial U}{\partial \theta_1}$ &c. being similarly obtainable.

We may also express a, b, c in terms of the components of stress $[\xi\xi], [\eta\xi]$ &c.; thus

$$\left. \begin{aligned} a &= -\frac{\partial}{\partial \xi} [\xi\xi] - \frac{\partial}{\partial \eta} [\eta\xi] - \frac{\partial}{\partial \zeta} [\xi\xi], \\ b &= -\frac{\partial}{\partial \xi} [\xi\eta] - \frac{\partial}{\partial \eta} [\eta\eta] - \frac{\partial}{\partial \zeta} [\zeta\eta], \\ c &= -\frac{\partial}{\partial \xi} [\xi\xi] - \frac{\partial}{\partial \eta} [\eta\xi] - \frac{\partial}{\partial \zeta} [\zeta\xi]. \end{aligned} \right\} . \quad . \quad (15)$$

In this section it has been virtually assumed that the strain-figure is exactly superimposed on the otherwise existing condition of the medium.

6. If the motion of the strain-figure is one of pure translation, $\omega_1, \omega_2, \omega_3$ are all constantly zero, and equation (10) reduces to

$$\left. \begin{aligned} F &= (11)\ddot{X} + (12)\ddot{Y} + (31)\ddot{Z}; \\ G &= (12)\ddot{X} + (22)\ddot{Y} + (23)\ddot{Z}; \\ H &= (31)\ddot{X} + (23)\ddot{Y} + (33)\ddot{Z}; \end{aligned} \right\} \text{at the same time} \quad . \quad (16)$$

Now construct the ellipsoid

$$(11)\xi^2 + (22)\eta^2 + (33)\zeta^2 + 2(23)\eta\zeta + 2(31)\xi\zeta + 2(12)\xi\eta = M\epsilon^2; \quad (17)$$

and it is evident from (16) above that when the motion is one of pure translation, the effective force (components F, G, H) is perpendicular to the diametral plane of the acceleration (components $\ddot{X}, \ddot{Y}, \ddot{Z}$), taken with respect to the ellipsoid (17).

(11), (22), (33), which are the sums of squares, are the values of the effective mass in the directions of the axes of reference; while (23), (31), (12), which are the sums of products, correspond to products of inertia in Rigid Dynamics, and may be called "products of mass." It may easily be shown that if r is the length of any given radius of the ellipsoid (17), the effective mass reckoned in the direction of this radius is $M\epsilon^2/r^2$. It may also be remarked that even when the motion is one of pure translation, there will in general be finite effective couples, as is immediately evident from (12).

7. *Case of Symmetry.*—Consider now the particular case in which the strain-figure is symmetrical about a point, and let this point be Ω , the origin of the axes of ξ, η, ζ . We shall then have

$$\begin{aligned} (11) &= (22) = (33) = \text{effective mass}; \\ (23) &= (31) = (12) = 0. \end{aligned} \quad \left. \vphantom{\begin{aligned} (11) &= (22) = (33) = \text{effective mass}; \\ (23) &= (31) = (12) = 0. \end{aligned}} \right\} \quad (17a)$$

It is also evident that a rotation of the strain-figure about any axis through its centre of symmetry corresponds to no physical change whatever; there is no possibility of such rotation, nor can any influence exerted on the medium have a tendency to turn the strain-figure about its centre of symmetry. Hence *a strain-figure symmetrical about a point is dynamically equivalent to a single particle of mass*

$$\begin{aligned} \iiint (\alpha_1^2 + \beta_1^2 + \gamma_1^2) d\xi d\eta d\zeta &= \iiint (\alpha_2^2 + \beta_2^2 + \gamma_2^2) d\xi d\eta d\zeta \\ &= \iiint (\alpha_3^2 + \beta_3^2 + \gamma_3^2) d\xi d\eta d\zeta \end{aligned}$$

placed at its centre of symmetry.

It is worth remarking that the theory now considered, though differing so widely from the theory of Boscovich, leads to the same dynamical results.

8. So far we have only considered the case of a single

strain-figure. When there are more than one, the experimental properties of matter lead us to suppose :—

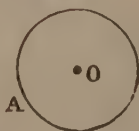
(i) That the entire distribution of displacement in the medium is to be found (at least very approximately) by compounding geometrically the distributions which the various strain-figures would have produced separately.

(ii) That the strain-figures exert forces upon one another, thus changing or tending to change their motion through the medium.

In speaking of the medium I have more particularly in mind the hypothesis of a turbulently moving liquid, as propounded by Sir W. Thomson* and modified by Prof. Fitzgerald†. Without entering on the subject mathematically (which I am unable to do), it may be noticed that, according to this view, the æther is made up of interlacing vortex-filaments, the interspaces between which are filled with quiescent or irrotationally moving liquid. The network of filaments is equivalent to a sponge-like compressible solid, whose pores, however, are completely filled with incompressible fluid. The medium, *taken as a whole*, is equivalent to an incompressible solid; but as the effectively liquid (*i.e.* irrotationally moving) portion of the medium cannot suffer any strain which calls up an opposing stress, it appears that in considering the strain-figure we should have to deal with the sponge-like compressible solid, made up of vortex-filaments.

9. *Gravitation.*—Disregarding this last speculation, let us return to the more general question of strain, and for simplicity take the case when the strain-figure is symmetrical about a point O. So far as this one strain-figure is concerned, we may, roughly speaking, divide the medium into two portions, in one of which the strain tends to increase, while in the other it tends to decrease. If these portions are separated by a single surface, then, in the present case of symmetry, this surface (A) will be spherical. On one side of A the medium will be so strained that a small additional strain of the same type would correspond to increased potential

Fig. 2.



* B. A. Report, 1887, p. 486; Phil. Mag., October 1887, p. 342.

† 'Nature,' May 9, 1889.

energy, while on the other side of A a small increase of strain would correspond to decreased potential energy. If the strain *outside* A corresponds to *increased* potential energy, two distant strain-figures will *repel* one another ; but if the strain *outside* A corresponds to *decreased* potential energy, two distant strain-figures will *attract* one another. If gravitation is to be explained in accordance with the theory of this paper, the above would seem to indicate the nature of the explanation. There must be some type of strain which would of itself be produced in the medium, were it not that in some other portion of the medium there would be an accompanying strain of a type corresponding to stability. Perhaps we may also vaguely infer why the gravitative attraction exerted by a body is proportional to its mass, for we have supposed both these quantities to be determined by the distribution of strain in the medium ; it seems possible, indeed, that the greater part of the effective mass of a strain-figure might reside in that region where attraction follows the Newtonian law.

10. *Collision*.—If we bring the centres of two strain-figures close together, so that the spheres A intersect sufficiently far, we shall be partly superposing two strain-distributions of such a type that the energy increases with the strain, so that repulsion may, perhaps, ensue. Of course this is only a very rough attempt at explaining the effects of collision, and takes no account of any *deformation* which might be produced in the strain-figures on bringing their centres to such close quarters. But even were such deformation produced, the results of a collision might still be of a very simple character. So long as the condition of § 4 is satisfied, *so long, that is, as the motions concerned are extremely slow compared with V (the velocity with which gravitation is propagated) the distribution of displacement in the medium will be at each instant the same as if the two strain-figures were at rest.* Hence, though the two strain-figures may recoil from one another with altered velocities they will not have acquired any vibratory motion from the impact. This statement is equally true whether the strain-figure is symmetrical or not, but in the case of symmetry about a point, the properties of a strain-figure will be identical with those of a Boscovich particle, exerting actions at a distance according to a law of force.

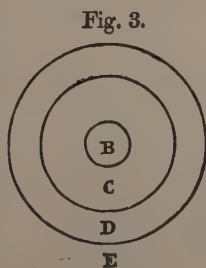
11. The result just established has a bearing on the dynamical theory of heat. We know that the number of degrees of freedom of a molecule, and consequently also of an atom, is probably finite. If an atom consisted of n strain-figures of the most general kind, the total number of degrees of freedom would be $6n$; but if each of the constituent strain-figures were symmetrical about a point the number would be reduced to $3n$. Since the spectra of elementary vapours teach us that the number of degrees of freedom, though finite, is usually very large, we are led to infer that most atoms are formed by the aggregation of a large number of strain-figures.

12. *Discrete Nature of Atoms.*—Why do the atoms form a discrete series, and why are all atoms of a given element identical in physical and chemical properties? These are questions which it would be difficult to answer, though it does not seem quite so difficult to point out a direction in which we may possibly look for an explanation. If an atom consists of a number of points endowed with inertia and with the power of exerting actions at a distance, it is hard to see how any explanation can be given: nor is the difficulty much less if the atom consists of a vortex ring or a vortex tangle, for we should then simply have to infer that the existing rings or tangles are as they are because they have always been so, and that an indefinite variety of rings or tangles of intermediate forms and sizes would be *à priori* perfectly possible. But if the atom consists of one or more strain-figures, the question becomes: Why do the strain-figures form a discrete series? Now we have conceived a strain-figure to be a disturbed condition of the medium, *which is of itself in stable equilibrium throughout*, and this immediately imposes an immense restriction on the possible varieties; since, also, the strains are impressed on a medium which would otherwise be homogeneous and isotropic, the conditions essential to stable equilibrium will be the same for all strain-figures, provided, that is, that the proximity of the centres of two or more does not disturb their form. But at this point a further assumption will be necessary; for if the turbulent motion or other structure of the medium were absolutely homogeneous (which implies infinite fine-grainedness), and if a strain-figure defined by a distribution of displacement

A were a possible one, then the figure defined by the distribution A magnified n diameters would be equally possible. Thus the possible strain-figures, though infinitely restricted in variety by the conditions of equilibrium, would form not a discrete but a continuous series, or possibly a discrete system of continuous series. We must suppose, then, that the *coarse-grainedness* of the medium has an influence in determining the *size* of possible strain-figures.

13. *Physical Illustration.*—A very imperfect illustration of this last point may be drawn from a physical phenomenon; for consider a region free from the action of gravity and filled with a saturated vapour, which by some means is maintained at constant temperature and pressure. If compression takes place, liquid will be formed, and will exist in equilibrium with the vapour, no intermediate condition of the substance being consistent with equilibrium and stability. But, neglecting surface-tension, the liquid need not be formed in masses of any special size; that is, the possible sizes of the drops of liquid will form a continuous series. If, however, surface-tension is taken into account, the case is different; for suppose that the vapour is *slightly supersaturated*, and contains a number of spherical drops of liquid. Very large drops will continue to increase, and very small drops will diminish, while drops of one particular size (if any such are present) will just be in equilibrium with the vapour, although the equilibrium is necessarily unstable. But notwithstanding that the analogy fails in this and in many other respects, it may serve to roughly illustrate the suggestion that the size of strain-figures, and consequently of atoms, is determined by the coarse-grainedness of the medium.

14. *Formation of Atoms.*—As regards the different varieties of atoms, we may conceive them to be made up of one kind or of several kinds of strain-figures. In place of the simpler form of fig. 2 we might for example imagine a form like fig. 3, where in the regions C, E the strain is such that the potential energy would decrease with increase of strain, while in the



regions B, D the strain and the potential energy would increase together. We might also imagine distributions which were not spherical, in which case it might happen (§ 4) that the laws of motion were less simple than those of Newton. It would not then necessarily follow that an atom consisting of more than one strain-figure would possess the same complex dynamical properties, though I am not aware of any evidence that the separate atoms or molecules of a substance move in accordance with Newton's laws.

Of course the operation described in § 2 is not intended to represent the *formation* of a strain-figure, but merely to show that the *existence* of such a distribution is conceivably possible. If the ultimate fluid had long ago possessed motion of the most general kind, we might imagine its present condition to be due to the degeneration of that motion into a fine-grained turbulence; and if, in the quasi-solid so constituted, the existence of strain-figures were possible, it seems not unlikely that such would incidentally have been formed, unless the motion fulfilled special conditions. I would suggest then, very tentatively, *that if the distribution of motion in the ultimate fluid had fulfilled certain special conditions, there would have been no atomic matter in the universe, and that the existence of matter as we know it is an indication that such conditions were wanting.*

Concerning the possibility of the "transmutation" of elements, this investigation leads to no immediate conclusion, but any conceivable superposition of two strain-figures would probably involve only finite potential energy, so that the effect of a very severe direct encounter might be to make the two strain-figures pass *through* one another. It seems possible, too, that at some stage of the impact the distribution might resolve itself into one or more strain-figures of a different kind, the entire effective mass not being *necessarily* the same as before.

15. It may be remarked that, according to the assumptions of this paper, every motion in the universe is ultimately due to *stationary motion* in a quasi-solid medium. This medium, which extends through all known space, is supposed to be (at least appreciably) perfect in its elastic properties, that is, free from viscous yielding and internal friction; and its ever

changing distribution of stress and strain is to be held accountable for all observed phenomena.

16. The subject of this paper being now explained, so far as the vagueness of my own views will allow, the fundamental proposition may be stated as follows :—

A given portion of matter consists, not of any individual portion of æthereal or other substance, but of modifications in the structure or energy or other qualities of the æther, and when matter moves it is merely these modifications of structure or of energy or of other qualities which are transferred from one portion of the æther to another.

The strain-figure has here been almost exclusively considered, and we have seen

(i) That provided the motion is slow compared with a certain velocity V (§ 4), a strain-figure will encounter no resistance in travelling through the æther, and will obey laws of motion which include Newton's Laws as a particular case.

(ii) That gravitative and inter-atomic forces may possibly be supposed to arise from the stresses which accompany the distribution of strain.

(iii) That a collision between two single strain-figures would not set them vibrating, so that an atom consisting of strain-figures would have a finite number of degrees of freedom, as required by the dynamical theory of heat.

(iv) That the size and nature of possible strain-figures, and therefore also of possible atoms, would be limited by the conditions of equilibrium, thus giving rise, perhaps, to a discrete series.

On the other hand, it is a special difficulty of my theory that we require some assumption as to the superposition of strain-figures to account for the fact that the mass of a material body is equal to the sum of the masses of its constituent particles.

In conclusion, I have to thank Mr. G. H. Bryan for his kindness in verifying some portions of the analysis, as well as for a suggestion which has immensely lessened the labour of calculation.

Note added November 21.

In the discussion which followed this paper, Professor

Fitzgerald mentioned that in his lectures he had spoken of matter as possibly travelling through space in the same way that a drop of water travels through a block of ice. The idea involved is the same as that which forms the basis of the foregoing pages. I have availed myself of Professor Fitzgerald's criticism to amend some remarks contained in the earlier part of the paper, and there is just one thing more which I should like to add. Every opinion expressed in the paper is not to be taken as an essential part of the theory, and, indeed, the most I can hope from an investigation so obviously incomplete is that it may prove suggestive to those who are working at the subject.

XXXI. *Struts and Tie-Rods with Lateral Loads.*

By Professor JOHN PERRY, D.Sc., F.R.S.*

I THINK that this subject has not yet been taken up scientifically; yet it is very important. The practical treatment of the whole subject of struts is in a very unsatisfactory condition; and it is mainly due to this that, of two bridges designed for the same spans and loads, by two engineers, one has sometimes more than twice the weight of the other; and in all probability the one of least weight is in some parts very much too strong, and in other parts has very little strength in excess of what is absolutely necessary.

It will be in the recollection of some of the members present that Professor Ayrton and I, in 1886, showed why experiment always gave a breaking-load for a strut which was less than that which results from Euler's theory.

A strut is a prismatic body of homogeneous material subjected to equal and opposite crushing forces at its ends. Taking its length, $2l$; the least moment of inertia of its cross section about a straight line through its centre of area, I ; Young's modulus of elasticity, E ; f_c the compressive stress which the material will stand: then for a strut hinged at its ends (that is, if the resultant force at each end acts at the centre of the end), Euler's theory gives the breaking-load as

* Read December 4, 1891.

the lesser of the two values

$$U = \frac{EI\pi^2}{4l^2}, \quad \dots \dots \dots (1)$$

and

$$f_c A. \quad \dots \dots \dots (2)$$

The length of strut for which these two answers are the same is

$$l = \frac{\pi k}{2} \sqrt{\frac{E}{f_c}},$$

if k is the least radius of gyration of the section about a line in the section through its centre of gravity.

Now we showed that want of homogeneity, inaccuracy of loading, and initial want of straightness in the strut all lead to a smaller load being able to break the strut, and, indeed, that an initial want of straightness (the assumption of an initial deflexion α at the middle) may be taken to represent all three kinds of discrepancy. We showed that for such values of l as make the two answers (1) and (2) nearly the same, a very small error α produces very great discrepancy—a very great diminution in the real breaking-load from Euler's breaking-load. We showed that when a strut is perfect and perfectly loaded, there is no deflexion until the breaking-load is reached; and that if a deflexion be artificially produced, the strut will straighten itself again, whereas in the imperfect strut there is a deflexion which increases at first in proportion to the load, and then more rapidly.

Such deflexions had been observed and carefully tabulated and published by Mr. Hodgkinson and Mr. Edwin Clarke, but hitherto no use had been made of them, and they had been regarded as unaccountable.

We pointed out that the error α in carefully made and carefully loaded struts seems roughly to follow a rule, and consequently that without making the unwarrantable assumption which has hitherto been used to make Gordon's formula appear to be a rational one, we had a right to look upon Gordon's formula as an empirical formula, which, being of the value (2) for short struts and of the value (1) for long struts, might be taken as right for struts of any length.

Since the publication of our paper ('The Engineer,' Dec. 10th, 1886) I have been in the habit of giving to students

problems on struts laterally loaded : as when a strut lies horizontally and is loaded with its own weight ; or as when a strut like the coupling-rod of a locomotive, or any connecting-rod, is loaded by its own centrifugal force.

In this paper I shall first give the general solution of such problems ; and it will include the old problem of want of straightness of the strut and inaccuracy of loading.

I take as the axis of x the straight line through the centres of area of the two ends of a strut. The origin is at the point of bisection of this line. The length of strut is $2l$. Systems of forces act at the ends such that their resultants are two equal and opposite pushing forces F (or pulling forces F_1) in the axis of x together with certain couples. Thus if a strut is hinged at the ends, and if the resultant forces do not act exactly through the centres of area, but at points whose distances are h (measured parallel to the plane in which bending is most likely to occur) from the centre of area, I take it that there are couples at the ends, of the amounts Fh or $-F_1h$. Again, if a strut is fixed at an end, I take it that there is an end couple $-M_0$ or $-M_1$ whose amount will be determined by given conditions of lateral loading and end push or pull. In this case M_0 and M_1 will include such terms as Fh . I can assume that the strut when unloaded is not quite prismatic, the ordinate of its centre line being y' . It is easy to make the treatment even more general by taking the initial shape a tortuous curve, and by considering bending as taking place in various planes ; but for nearly all practical purposes it is sufficient to deal with one plane only.

I suppose the strut to have a lateral loading such that if it were a beam *supported* merely at the ends, the bending-moment would be $\phi(x)^*$.

It can readily be shown that the total bending-moment at

- * If the strut is loaded uniformly laterally with a total load W ,

$$\phi(x) = \frac{1}{2} W(l^2 - x^2).$$

If the strut has a single load at the middle there is a discontinuity in the function at the middle ; but we may put it

$$\phi(x) = \frac{1}{2} W(l - \sqrt{x^2}),$$

the minus sign being taken whether x is positive or negative. There are easy graphical methods, well known to all practical engineers, for finding $\phi(x)$ for any system of lateral loading however complicated.

any section of a strut or tie-rod, if $-M_0$ and $-M_1$ are the couples at its ends, is

$$Fy + \phi(x) - \frac{1}{2}(M_0 + M_1) + x \frac{M_0 - M_1}{2l}, \quad \dots \quad (3)$$

or

$$-F_1y + \phi(x) - \frac{1}{2}(M_0 + M_1) + x \frac{M_0 - M_1}{2l}. \quad \dots \quad (4)$$

The curvature being always small is $\pm \frac{d^2y}{dx^2}$, and we take the + or - sign according to the sign given to bending-moment, and hence we have generally for struts

$$Fy + \phi(x) - \frac{1}{2}(M_0 + M_1) + x \frac{M_0 - M_1}{2l} = -EI \left(\frac{d^2y}{dx^2} - \frac{d^2y'}{dx^2} \right),$$

or

$$\frac{d^2y}{dx^2} + \frac{F}{EI}y = \frac{d^2y'}{dx^2} - \frac{1}{EI}\phi(x) + \frac{1}{2EI}(M_0 + M_1) - \frac{x}{EI} \frac{M_0 - M_1}{2l}; \quad (5)$$

and for tie-rods we have merely to write $-F_1$ instead of F .

I shall assume for the present that EI is constant everywhere.

Now $\phi(x)$ can always be developed in a Fourier's series. But it will sometimes be found convenient to express it in the form

$$\begin{aligned} \phi(x) = & a_0 + rx + tx^2 + a_1 \cos sx + b_1 \sin sx + \\ & a_2 \cos 2sx + b_2 \sin 2sx + \&c. + a_i \cos isx + b_i \sin isx + \&c., \end{aligned}$$

where $s = \frac{\pi}{2l}$.

Terms in x^3 , x^4 , &c., may easily be taken also.

We may put this symbolically as

$$\phi(x) = a_0 + rx + tx^2 + (a_i, b_i, is). \quad \dots \quad (6)$$

Whatever y' may be, we may express it symbolically as

$$y' = \alpha_0 + (\alpha_i, \beta_i, is). \quad \dots \quad (7)$$

Hence (5) may be written:—

For struts:

$$\begin{aligned} \frac{d^2y}{dx^2} + \frac{F}{EI}y = & -\frac{a_0}{EI} + \frac{M_0 + M_1}{2EI} - \frac{x}{EI} \left(r + \frac{M_0 - M_1}{2l} \right) \\ & - \frac{t}{EI}x^2 - \left(i^2s^2\alpha_i + \frac{a_i}{EI}, i^2s^2\beta_i + \frac{b_i}{EI}, is \right); \quad \dots \quad (8) \end{aligned}$$

and for tie-rods we have the same equation, if $-F_1$ be substituted for F .

Now if n^2 be written instead of $\frac{F}{EI}$ in the case of a strut, and instead of $\frac{F_1}{EI}$ in the case of a tie-rod, we have the following solutions.

For a Strut :—

$EIy = M \cos nx + N \sin nx + A_0 + Rx + Tx^2 + (A_i, B_i, i s), \quad (9)$
where M and N are arbitrary constants,

$$A_0 = \frac{-a_0 + \frac{1}{2}(M_0 + M_1)}{n^2} + \frac{2t}{n^4}.$$

$$R = -\frac{r}{n^2} + \frac{M_1 - M_0}{2ln^2}.$$

$$T = -\frac{t}{n^2}.$$

$$A_i = -\frac{i^2 s^2 \alpha_i EI + a_i}{n^2 - i^2 s^2}.$$

$$B_i = -\frac{i^2 s^2 \beta_i EI + b_i}{n^2 - i^2 s^2}.$$

For a Tie-rod :—

$EIy = Me^{nx} + Ne^{-nx} + A_0' + R'x + T'x^2 + (A_i', B_i', i s), \quad (10)$
where

$$n = \sqrt{\frac{F_1}{EI}},$$

where M and N are arbitrary constants,

$$A_0' = \frac{2t}{n^4} + \frac{a_0 - \frac{1}{2}(M_0 + M_1)}{n^2},$$

$$R' = \frac{r}{n^2} - \frac{M_1 - M_0}{2ln^2},$$

$$T' = \frac{t}{n^2},$$

$$A_i' = \frac{a_i + \alpha_i EI i^2 s^2}{n^2 + i^2 s^2},$$

$$B'_i = \frac{b_i + \beta_i EI i^2 s^2}{n^2 + i^2 s^2}.$$

When y is known, the greatest value of the bending-moment is known. A line through the centre of a section at right angles to the plane of bending is always called the neutral line of the section, even in cases like the present where the name is a little misleading; and if z_c and z_t are the greatest distances of points in the section from the neutral line, on the compressive and on the tensile sides respectively, and if μ is the greatest bending-moment, then in struts, if A is the area of cross section,

$$\frac{\mu}{I} z_c + \frac{F}{A} = f_c, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{\mu}{I} z_t - \frac{F}{A} = f_t; \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and in tie-rods,

$$\frac{\mu}{I} z_c - \frac{F_1}{A} = f_c, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\frac{\mu}{I} z_t + \frac{F_1}{A} = f_t, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where f_c and f_t are the maximum compressive and tensile stresses to which any part of the strut or tie-rod is subjected. In many practical cases $z_c = z_t$, and then I/z_c is called Z , the strength-modulus of the section.

We can therefore find the strength and stiffness of any strut of uniform section fixed or free at either or both ends when loaded laterally in any way whatever.

When, as in many practical cases, it is allowable to take an approximate value for $\phi(x)$, and especially when the lateral loading and initial shape are such that the strut is symmetrical about the axis of y , the work can be greatly simplified by not introducing such terms as r and t . In the symmetrical cases such terms as b_i and β_i are absent from the expressions, and $M_0 = M_1$. If the ends are not fixed, $M_0 = M_1 = -Fh$.

Examples.

I. A uniform straight strut has a lateral load W uniformly distributed. It will be found that in this case we have very nearly

$$\phi(x) = \frac{1}{4} W l \cos \frac{\pi}{2l} x. \quad \dots \quad (15)$$

Then, if F is the resultant of the pushing forces, (8) becomes

$$\frac{d^2 y}{dx^2} + \frac{F}{EI} y = \frac{M_0}{EI} - \frac{1}{4} \frac{W l}{EI} \cos \frac{\pi}{2l} x. \quad \dots \quad (16)$$

Solving this by the rule given above and applying the conditions that $y=0$ when $x=l$, and $\frac{dy}{dx}=0$ when $x=0$, it is easy to see that the solution becomes

$$y = \frac{M_0}{F} - \frac{M_0}{F \cos l \sqrt{\frac{F}{EI}}} \cos x \sqrt{\frac{F}{EI}} + \frac{\frac{1}{4} W l}{EI \frac{\pi^2}{4l^2} - F} \cos \frac{\pi}{2l} x. \quad (17)$$

We can at once work out from this, cases in which with such a lateral loading we have any equal couples M_0 applied at the ends, including the case in which the end is hinged, but the resultant push at the ends is applied with the inaccuracy h , in which case $-M_0 = Fh$. In case the strut is fixed at the ends we apply the condition $\frac{dy}{dx}=0$, when $x=l$. This gives us

$$M_0 = \frac{\pi W \sqrt{FEI}}{8(U-F)} \cot l \sqrt{\frac{F}{EI}}.$$

It is not instructive to pursue this example unless numerical values are taken for the dimensions of the strut and the loading W . If the strut is hinged at the ends and $h=0$, then $M_0=0$, and

$$y = \frac{\frac{1}{4} W l}{EI \frac{\pi^2}{4l^2} - F} \cos \frac{\pi}{2l} x. \quad \dots \quad (18)$$

As this is a simple case, common in practice, I will work out the stresses.

(18) is evidently true even when $F=0$. The deflexion in the middle is

$$y_1 = \frac{\frac{1}{4} Wl}{EI \frac{\pi^2}{4l^2} - F}; \quad \dots \quad (19)$$

and the greatest bending-moment μ is

$$\mu = Fy_1 + \frac{1}{4} Wl,$$

or

$$\mu = \frac{1}{4} Wl \left(\frac{\frac{EI\pi^2}{4l^2}}{\frac{EI\pi^2}{4l^2} - F} \right). \quad \dots \quad (20)$$

If $W=0$ and if μ has any value whatever, the denominator must be 0. Putting it equal to 0, we have Euler's law for the strength of struts which are so long that they bend before breaking. If Euler's value of F be called U ,

$$U = \frac{EI\pi^2}{4l^2}. \quad \dots \quad (21)$$

Using this as a symbol, (20) becomes

$$\mu = \frac{1}{4} Wl \frac{U}{U - F}; \quad \dots \quad (22)$$

and we can at once apply (11) and (12) to find the greatest compressive and tensile stresses in the strut. Then, writing $\frac{U}{A}$ as β , and $\frac{F}{A}$ as ω , we see that if f is the compressive stress which the material will bear,

$$\left(1 - \frac{\omega}{f}\right) \left(1 - \frac{\omega}{\beta}\right) = \frac{Wl}{4fZ}. \quad \dots \quad (23)$$

From this, if we are given the dimensions of the strut and f and W , it is easy to calculate ω . The solution of the quadratic is

$$2\omega = f + \beta - \sqrt{(f + \beta)^2 + \left(\frac{Wl}{Z} - 4f\right)\beta}, \quad \dots \quad (24)$$

the *minus* sign being taken because ω is evidently less than f and less than β .

Coupling-Rods.—Starting from this result, my students have for several years worked out the relative breadths and

thicknesses of the sections of coupling-rods and connecting-rods of engines, which are struts whose lateral loads are due mainly to centrifugal force.

Every point in a coupling-rod describes a circle of radius r inches. If the section is, say, rectangular (an elliptic section is just as easy to deal with) of dimensions d in the plane of motion and b at right angles to the plane of motion; taking the whole mass as $2lb d \times .28$ divided by 32.2 ; the centrifugal force

$$W = \frac{lbdrn^2}{62940} \text{ lbs. wt.}$$

if the rod makes n revolutions per minute.

In one direction a coupling-rod is a strut hinged at the ends, and the thrust per square inch that it can receive, assuming that it is properly made, is to be the same as the thrust which it will stand in the other direction. In the first direction it has lateral loading due to centrifugal force.

Now it might be supposed that in the second direction the strut is as if fixed at the ends; but the pins are very short, and the resultant load is certainly not applied axially, and it is safer to assume that in this direction also the strut is as if hinged at the ends. I assume this, although perhaps from the possibility of accidental untrue loading we ought to go further in this safe direction and assume that the strength of the strut is less. Assuming it hinged also in the second direction, however, the thrust per square inch which it can receive is $EI_1 \frac{\pi^2}{4l^2} / bd$, and in this case $I_1 = \frac{db^3}{12}$; so that, if $E = 3 \times 10^7$,

$$\omega = 6.17 \frac{b^2}{l^2} \times 10^6.$$

This is the value to use in formula (23).

Taking f for the steel used as 20,000 lb. per square inch, being the proof stress which the material will stand for an infinite number of reversals of stress, we find for equal strength in the two directions, applying (23), that

$$8.4 \times 10^8 \left(1 - 308 \frac{b^2}{l^2}\right) \left(1 - \frac{b^2}{d^2}\right) = n^2 l^2 r / d.$$

Thus, for example, if $b=1$, and $l=30$, and $r=12$, we have the following tables:—

If $b=1$.

d .	n .
1	0
1.5	205
2	277
2.5	327
3	368
4	437
6	545

If $b=1.5$.

d .	n .
1.5	0
2	125
2.5	170
3	202
4	249
6	318
10	440

To illustrate these results. On a certain locomotive engine the coupling-rods are 68 inches long or $l=34''$, $d=3\frac{1}{2}$, $b=1\frac{3}{8}$, $r=12$.

Now it will be found that this rod is equally strong in the two directions if $n=298$ revolutions per minute. The limit of speed which has practically been settled for the engine by the breaking of similar coupling-rods is 258 revolutions per minute.

Without trying to make a rod equally strong to resist bending in the two directions, it is interesting to consider the F or rather ω which a coupling-rod will stand when revolving at n revolutions per minute.

Using (24) and taking

$$E=3 \times 10^7, \quad \frac{Wl}{Z} = \frac{l^2 r n^2}{10490d},$$

$\beta=6.17 \times 10^6 \frac{d^2}{l^2}$, so that ω does not depend upon b .

If $l=30$ and $r=12$, and $f=20,000$, we find for various values of d and n the following values of ω :—

Values of n .	ω if $d=2$.	ω if $d=3$.	ω if $d=4$.
0	20000	20000	20000
100	16720	18810	19090
200	11760	15455	16930
300	5505	10690	13400
350	2615	7880	11150
394	0
400	5070	8800
450	2045	6150
483	0
500	3400
557	0

Taking as before $l=30$ and $r=12$, but now $f=10,000$, we find :—

Values of n .	ω if $d=2$.	ω if $d=3$.	ω if $d=4$.
0	10000	10000	10000
100	8166	8995	9500
200	3976	6185	8450
250	1490	4240
279	0
300	2010	4200
340	0
350	2050
394	0

It is quite easy to make similar calculations on sections of **I** shape. Rods of this section may be made equally strong to resist bending in the two directions, at much higher speeds than are possible in the case of the rectangular section. The elliptic section is not so good as the rectangular, and the **I** section is better than either.

II. A uniform straight tie-rod has a lateral total load W uniformly distributed. The resultant of pulling forces at the ends is F_1 and passes exactly through the centres of the ends ; (8) becomes

$$\frac{d^2y}{dx^2} - \frac{F_1}{EI}y = + \frac{M_0}{EI} - \frac{\frac{1}{4}Wl}{EI} \cos \frac{\pi}{2l}x. \quad (25)$$

Using n^2 for $\frac{F_1}{EI}$,

$$y = Me^{nx} + Ne^{-nx} - \frac{M_0}{n^2EI} + \frac{\frac{1}{4}Wl \cos \frac{\pi}{2l}x}{EI\left(n^2 + \frac{\pi^2}{4l^2}\right)}. \quad (26)$$

Applying the conditions $y=0$ when $x=l$, and $\frac{dy}{dx}=0$ when $x=0$, we find

$$y = \frac{M_0}{EI n^2} \left\{ \frac{e^{nx} + e^{-nx}}{e^{nl} + e^{-nl}} - 1 \right\} + \frac{\frac{1}{4}Wl \cos \frac{\pi}{2l}x}{EI\left(n^2 + \frac{\pi^2}{4l^2}\right)}. \quad (27)$$

If we suppose that no couples are applied at the ends, or $M_0=0$,

$$y = \frac{\frac{1}{4}Wl \cos \frac{\pi}{2l}x}{F_1 + \frac{\pi^2}{4l^2}EI} \quad \dots \quad (28)$$

This is identical with (18) if $F_1 = -F$. So that in this simple case we can use the same expression for the shape, whether the bar is a strut or a tie-rod.

In the case of a strut,

$$\frac{\frac{1}{4}Wl}{Z} \frac{U}{U-F} + \frac{F}{A} = f_c, \quad \dots \quad (29)$$

$$\frac{\frac{1}{4}Wl}{Z} \frac{U}{U-F} - \frac{F}{A} = f_t, \quad \dots \quad (30)$$

and by taking F in the expressions negative, we find f_c and f_t in a tie-rod.

A numerical example will illustrate these formulæ. I will take a round rod of mild steel lying in a horizontal position, so that it is loaded laterally by its own weight, dimensions being in inches. If d is its diameter, the weight of 1 cubic inch of material is 0.28 lb. and $E=3 \times 10^7$,

$$I = \frac{\pi d^4}{64} \quad \text{and} \quad Z = \frac{\pi d^3}{32}, \quad U = 3.6 \times 10^6 \frac{d^4}{l^2},$$

$$f_c \text{ or } f_t = \frac{4.03 \times 10^6 d^3}{3.6 \times 10^6 \frac{d^4}{l^2} - F} \pm \frac{F}{\frac{\pi}{4}d^2};$$

the *plus* sign being taken in calculating f_c and the *minus* in calculating f_t . Also

$$y_1 = \frac{.11d^2l^2}{3.6 \times 10^6 \frac{d^4}{l^2} - F}.$$

Take $d=1$ inch, $l=60$ inches,

$$f_c \text{ or } f_t = \frac{4.03 \times 10^6}{1000 - F} \pm 1.273F \quad \text{and} \quad y_1 = \frac{396}{1000 - F}.$$

From this I have calculated the following table. The negative values of F indicate that instead of a strut we have a tie.

F .	f_c .	f_t .	y_1 .
1000	∞	∞	∞
900	41446	39154	3.96
800	21168	19132	1.98
700	14324	12542	1.32
500	8697	7423	0.792
300	6648	4866	0.566
100	4605	4351	0.440
0	4030	4030	0.396
- 100	3537	3791	0.360
- 300	2718	3482	0.305
- 500	2031	3323	0.264
- 700	1479	3261	0.233
- 900	975	3267	0.208
-1200	304	3360	0.180
-1348	0	3432	0.169

We see, then, that a comparatively small thrust F produces excessive stress in the strut and great deflexion y_1 .

We see also that a tensile force F_1 of 900 lb. halves the deflexion of the unpulled beam, and not only reduces the compressive stress to less than a quarter of its old amount, but it has also reduced the tensile stress by nearly 20 per cent. And a tensile load F_1 of 1348 lb. completely destroys all compressive stress in the strut, whilst reducing the greatest tensile stress of the unpulled strut by 15 per cent.

In fact when a beam is very long, pulling out its ends may reduce the compressive stress in it to nothing, whilst also reducing the tensile stress. A longer or smaller beam than that which I have taken will illustrate this matter even better.

The tension F_1 which will make f_t a minimum is

$$F_1 = \sqrt{\left(\frac{AWl}{4Z} U\right)} - U,$$

and when $\frac{AWl}{4Z}$ is so small as U then the unpulled beam itself has the minimum tensile stress.

If $\frac{AWl}{4Z}$ is less than U it requires a pushing force F to give the minimum tensile stress.

In all cases there is, of course, a pulling force F_1 which will cause the beam to have no compressive stress in it. In some cases there is a pushing force F which will cause the beam to have no tensile stress in it, but this can only be the case when

$$U \text{ is not less than } \frac{WlA}{Z}.$$

In this case, a beam to carry lateral load may be built up of separate blocks of material which have just sufficient friction to prevent slipping due to shearing forces at the joints. It is a case which may be treated graphically by Professor Fuller's Method, as the change of shape is small.

It will be seen from the above simple example that very instructive numerical problems may be given. Thus if, instead of the lateral loading, we have merely couples, $-M_0$ or m , applied to the ends of a strut, we use $W = 0$ in (17).

When $x=0$ we have the greatest value of y_1 , or

$$y_1 = \frac{m}{F} \left\{ \frac{1}{\cos l \sqrt{\frac{F}{EI}}} - 1 \right\}.$$

Also $Fy_1 + m = \mu$ the greatest bending-moment in the strut,

or
$$\mu = \frac{m}{\cos l \sqrt{\frac{F}{EI}}},$$

$$f_c = \frac{m}{Z \cos l \sqrt{\frac{F}{EI}}} + \frac{F}{A},$$

$$f_t = \frac{m}{Z \cos l \sqrt{\frac{F}{EI}}} - \frac{F}{A}.$$

Taking for numerical example a round bar of steel as

before, and taking $d=1$ inch, $l=48$ inches, and writing the angle in degrees to simplify calculation, we have

$$f_c \text{ or } f_t = \frac{10 \cdot 2m}{\cos 2 \cdot 267 \sqrt{F}} \pm 1 \cdot 273F,$$

$$y_1 = \frac{m}{F} \left\{ \frac{1}{\cos 2 \cdot 267 \sqrt{F}} - 1 \right\}.$$

If $F=0$, $f_c \text{ or } f_t = 10 \cdot 2 m$, $y_1 = \frac{m l^2}{2EI}$ or $7 \cdot 821 \times 10^{-4} m$,

Euler's strength or U for this strut is 1577 lb. It is evidently obtainable by putting the quantity inside the bracket equal to infinity.

First take $F=1500$ lb. Then

$$y_1 = \frac{m}{60}, \quad 2 \cdot 267 \sqrt{F} = 87^\circ 80,$$

$$f_c \text{ or } f_t = 266m \pm 1910.$$

m .	f_c .	f_t .	y_1 .	$f_c \text{ or } f_t$, if $F=0$.
0	1910	-1910	0	0
1	2176	-1644	0167	10.2
2	2442	-1378	0333	20.4
3	2708	-1112	0500	30.6
5	3240	-580	0833	51.0
7.18	3820	0	1197	73.2
10	4570	750	1667	102
20	7230	3410	3333	204
80	23190	19370	1.333	816

So that an endlong load only sufficient by itself to produce a stress of 1910 lb. per sq. inch and a bending-moment which by itself would only produce a stress of 816 lb. per sq. inch, if both act together, produce a stress of 23190 lb. per sq. inch. In fact, the stress due to m alone is intensified more than 26 times.

Similarly with an endlong thrust of 1400 lb. the stress due to the lateral bending-moment m is intensified 11.04 times. With an endlong thrust of 1000 lb. the stress due to m is intensified 3.185 times. With an endlong thrust of 500 lb. the stress due to m is intensified 1.58 times.

It is obvious that in any strut the stress due to m , a lateral bending-moment, is more and more intensified as F approaches Euler's load U . And if m is due to an inaccuracy of loading, h , the effect becomes four times as noticeable in struts of half the length.

From the above example one sees that it was quite possible for a strong man like Samson to exert a sufficient lateral force to produce fracture in the columns of the temple at Gaza.

Professor Fuller's graphical method of dealing with a metal arched rib is not applicable to struts unless we assume $Fy=0$, and this is of course a very wrong assumption. For the same reason, the graphical method cannot be applied to very flat arches, because it assumes that we know the shape of the loaded arch. If Professor Fuller's method were applicable, we could at once deal graphically with a strut of varying section. This case does not usually need to be solved in practice, but if it must be solved the following method will do.

1st. Assume that EI is constant and of its average value, and obtain the shape of the strut as in (10).

2nd. Use this value of y in the term $\frac{F}{EI}y$ of (8); and use the proper value of I expressed as a function of x , and then by mere integration of (8) find the more correct value of y .

By repeating this process we can obtain y more and more accurately.

Problem.

When a strut is loaded laterally (say uniformly), say that it is hinged at the ends; it is possible by applying the endlong thrust F untruly, that is making the resultant thrust pass through a point h inches from the centre of each end, to give such a value to h as will enable the strut to withstand a maximum load F . In this case it is obvious that the greatest compressive stresses are:—

at the ends, $f = F\left(\frac{h}{Z} + \frac{1}{A}\right)$;

$$\text{at the middle, } f = \frac{F}{A} + \frac{\frac{1}{2}Wl}{Z} \frac{U}{U-F} - \frac{Fh}{Z \cos l \sqrt{\frac{F}{EI}}}.$$

The problem to find for a given strut and given maximum stress f the value of h which will cause F to be a maximum is easily worked out if a numerical example be taken.

It is obvious from this and from other examples that may be taken, that whether a strut is loaded laterally or is imperfectly straight or has other faults which would cause it to break with a load F smaller than U and fA , it is always possible to adjust the loading so that F may be increased. Thus if a strut is being tested and if for small loads we find that it deflects, the load ought to be readjusted in such a way that its application produces the minimum deflexion; and now if the load is increased we shall find that the strut breaks with a much greater load than if we had not adjusted the method of its application. This adjustment of the method of loading ought to be applied also in the building of structures.

It will be seen that I have merely touched on a few of the many very interesting problems on structures to which the general treatment (3) to (30) may be applied. The lateral load required to prevent a straight strut from bending and to cause its breaking-thrust to be fA is evidently small. We can assume, however, an initial want of straightness or an accidental deflexion, and so find how very slight need be the diagonal bracing in the large struts used in bridge-building.

I have to thank Messrs. Field and Clinton, of the Finsbury College, for checking the mathematical work in this paper.

XXXII. *On a Permanent Magnetic Field.*

By W. HIBBERT, A.I.E.E., F.I.C.*

IN the electrical laboratory of the Polytechnic Institute, Regent Street, the earth's magnetic field varies so much that it cannot be assumed as a basis for reasonably accurate measurement. For this reason it was formerly my habit to

* Read December 4, 1891.

give students a certain bar-magnet as a temporary standard, the number of lines passing out of the bar being determined afresh from time to time. This magnet was an old one, and its varied experience (now described under the term "ageing") had given it an approximate constancy. The constancy was, in fact, good enough to suggest the possibility of getting a really permanent magnet. I therefore made some tentative efforts to ascertain the effect of slight variations in the hardness and temper of steel on its magnetic properties, with the result that I found the subject too great for my resources.

But the growth of the modern idea of a magnetic circuit suggested the possibility of achieving my purpose in a way that would be largely independent of the peculiarities of different brands of steel, as well as of the various physical conditions caused by differences in tempering.

I therefore provided a short straight bar-magnet with a couple of arched pole-pieces, of such length and sectional area that there was left between them a narrow air-gap of very small magnetic "resistance." A very flat coil of wire thrust in or out of this gap gave electromagnetic impulses whose value was fairly constant. Experience with these simple instruments led me to believe that a more rigorous application of the principle would give a truly permanent magnetic field. How far this has been realized will appear from the considerations which I now have the honour to submit to the Society.

The first design that at all approximated to a closed magnetic circuit consisted of a cylindrical steel rod with hemispherical pole-pieces. For reasons into which I need not go, this has been superseded by the following modification.

Fig. 1 is a general view of the instrument. Fig. 2 is a vertical section of the magnetic circuit. In this, NS is a cylindrical steel rod, $2\frac{1}{4}$ inches long and 1 inch in diameter, attached to two cast-iron pole-pieces. The upper pole-piece, *aa*, is a circular disk, 4 inches in diameter and $\frac{5}{8}$ inch thick, carefully bored to fit the upper end of NS. The lower pole-piece, *bb*, is nearly hemispherical, and about $\frac{1}{2}$ inch thick in the wall. This is attached to the steel rod by means of an iron pin, let into NS and screwed into *bb*.

The opposed faces of the pole-pieces are carefully turned in

the lathe, so as to leave between them a circular air-gap less than $\frac{1}{16}$ inch wide.

Fig. 1.

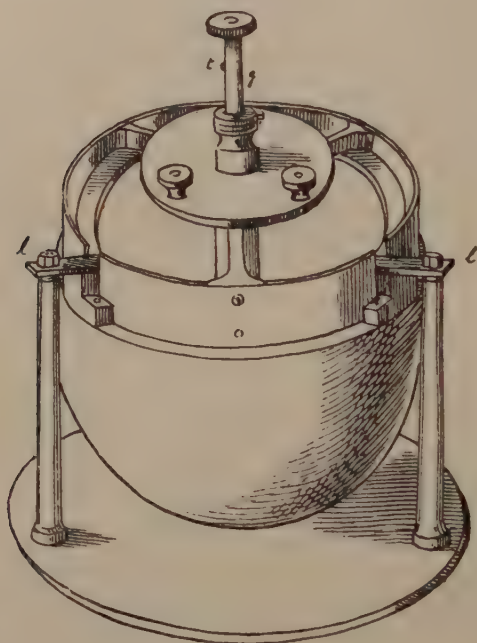
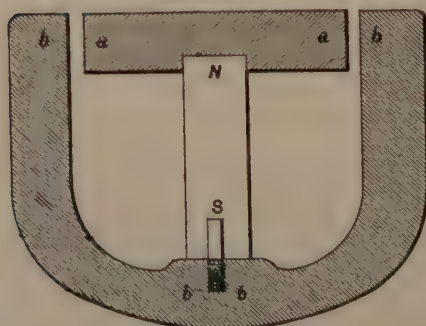


Fig. 2.



The above description gives the form and dimensions of the magnetic part of the instrument. It is magnetized after being put together by a current sent through a coil wound on the steel rod.

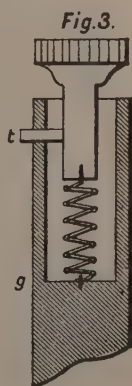
The other parts consist of:—

(1) Mechanical arrangements which will rigidly preserve the pole-pieces in position.

(2) Arrangements which will carry and guide a coil of wire as it is thrust through the field in the very narrow air-gap. Screwed to the upper surface of the disk *aa* is a brass casting with three projecting lugs *ll* (fig. 1). These lugs are screwed both to small blocks attached to the hemispherical pole-piece, and also to the tops of three brass pillars (fig. 1).

The coil is wound in a shallow groove cut on the outside of a brass ring, which is turned thin enough in the wall to slide freely through the air-gap. Attached to its upper edge are three arms which radiate from a central boss, the latter sliding up or down on the central guide-rod *g* (fig. 1). The radial arms support an ebonite disk on which are fixed the terminals of the coil.

At the upper end of the guide-rod *g* is an arrangement for allowing the ring to fall whenever the experimenter desires. The upper end of *g* is bored out (fig. 3), and a horizontal slot is cut in the thin wall thus made. In the inner space and connected with the milled head is a rod, from the side of which projects a tooth *t* passing through the slot, whilst from the lower end passes a spiral spring also fixed into the body of *g*. The tooth *t* is thus kept in a definite position, from which it can be moved aside either by a rotation of the milled head, or by the upward movement of the boss connected with the ring. On the inner edge of the boss is a slot through which *t* can pass, but cut slightly on one side of the zero position of the tooth. When an electromagnetic impulse is desired the coil is raised so that the boss, and therefore the ring, rests on *t*. The position of *t* and the height of the ring are so chosen that when the ring is at rest on *t* the coil is altogether above the gap. By a simple rotation of the milled head the coil can be made to fall through the field at any desired moment. The electromagnetic impulse then given to the circuit is, of course, equal to the number of lines inter-



linked with the coil during the fall, multiplied into the number of turns of wire in the coil.

Three instruments of the type just described have been made and tested. The means adopted for testing are not the best that could be specified for the purpose. They were chosen because they give reasonable accuracy, and yet allow of a fair number of observations being taken in a limited time.

The method was to compare two throws of a galvanometer-needle, one produced by the discharge of a condenser, the other produced by the magneto-inductor.

The condenser employed was a mica condenser of 0.333 microfarad capacity. The potential difference for charging was obtained from four accumulators, whose electromotive force was determined by comparison with a Latimer-Clark cell, the comparison being made by the potentiometer method.

Having first taken a fair number of observations from the condenser, a corresponding number were taken from the magneto-inductors, the resistance in circuit with each being adjusted till the throw was practically the same as that obtained from the condenser.

I decided on this because it allowed me to use an ordinary damped reflecting-galvanometer. The object being simply to test constancy, it was evident that the subsequent tests would be very much like the first, and that by the above method of working I might neglect damping, and also any consideration of the law of the galvanometer deflexion.

As the observations run through summer and winter they had to be corrected for temperature variation. The coefficients used were those now generally accepted, namely:—For the Latimer-Clark cell 0.077 per cent. per 1°C ; for the copper wire of the galvanometer 0.38 per cent. per 1° ; and for the German-silver resistances 0.044 per cent. per 1° .

It now remains to give the results obtained. As my present purpose is simply to show how far constancy of field has been obtained, it is not necessary to give many details. Before giving the results it will be proper to state the chief differences between the instruments numbered I., II., and III. in the Table.

Hemisphere No. I.—Air-gap a little more than $\frac{1}{16}$ inch and not quite uniform. Magnetized by flashing current from four accumulators, July 29, 1890. Heated and cooled several times on different days. Very small decay to August 11, on which day the number of lines = 21,007. Next day, August 12, 1890, lines = 21,120. Seven months later, March 14, 1891, lines = 21,035. Accident to coil caused me to take to pieces and magnetize afresh. A current flashed through at a temperature of about 50°C.

Hemisphere No. II.—Most highly finished. Air-gap rather less than $\frac{1}{16}$ inch. Magnetized by flashing. Lines about 34,000. Nearly 50 per cent. greater than in I. This showed tendency to fall. I therefore adopted the method known as “reducing,” by sending reverse current through magnetizing coil. Reduction of field about 5 per cent.

Hemisphere No. III.—Magnetic system not so well supported as in two previous instruments. Magnetized by flashing current, and then “reduced” about 20 per cent.

Date.	Temperature of Inductor.	Lines in Inductor Field.		
		No. I.	No. II.	No. III.
April 16, 1891.....	50 C.	22,030		
“ “ “	20	21,790		
“ 22 “	12.5	21,730		
“ 23 “	13.5	21,710	32,360	
May 8 “	16.5	21,710	32,420	
“ 23 “	13	21,680	32,330	
“ 27 “	13.5	29,140
“ 30 “	16	21,720	32,410	29,290
June 6 “	18	21,720	32,380	29,270
“ 12 “	22	21,780	32,470	29,260
“ 29 “	21.5	21,720	32,345	29,290
July 10 “	19	21,790	32,510	29,500
“ 27 “	20	21,700	32,470	29,550
“ 31 “	17.5	21,780	32,460	29,530
Sept. 22 “	16	21,690	32,400	29,470
Nov. 10 “	13	21,700	32,400	29,480
Density of lines in air-gap per sq. cm. }	515	770	700

The figures in the column headed "No. of Lines" are obtained by the expression

$$N = \frac{100 CVR}{t} \cdot \frac{d'}{d};$$

where C = capacity of condenser in microfarads,

V = potential difference in volts,

R = resistance in circuit with magneto-coil,

t = number of turns of wire on magneto-coil,

d', d = throws from magneto-inductor and condenser respectively.

The factor 100 translates from practical to absolute units.

In discussing the figures it must be remembered that high accuracy was not attainable at the time of working. The observations were all made in comparatively brief intervals of leisure, and it is likely that the probable error is about 1 in 300 or 400.

There is practically no evidence of magnetic decay in seven months. Such small changes as are indicated point the other way, but I am inclined to attribute most of them to slight inaccuracy of the temperature-correction for the resistances of the circuit.

The two instruments that were reduced by a demagnetizing current (Nos. II. and III.) show a tendency to rise. This tendency is most marked in the instrument which was reduced most. It is evident that only a slight reduction is to be allowed.

No. I., which was magnetized warm, but not "reduced," fell while cooling, and then showed a very slight tendency to increase. It is better than the other two, and is constant enough to be used for any purpose for which an earth inductor is employed.

Nothing is here said of the temperature variation of this magnetic field*. I have made some observations on that point, and found it very small, but as the tests were made

* Magnetic field is of course not the right name. The instrument raises once again the question of naming the quantity sometimes called "total field," "total induction," or so many "gausses."

with hastily-adjusted appliances, I have thought it better to reserve the matter for further investigation. The temperature range involved in the Table is 10°C. , but the figures cannot be used for deducing the temperature-coefficient because of the other sources of error. I am inclined to think that the temperature variation of the condenser (which has not been applied to the calculations) practically neutralizes that of the inductor.

In addition to the evidence of constancy afforded by the Table, there are in my note-books several facts which point the same way. Of these I shall mention only one.

It is well known that magnetic decay is most pronounced just after magnetization, especially if the magnet is subjected to vibration. In several early cases I tested the effect of vigorous blows during and immediately after magnetization, but the evidence of loss was generally very feeble, and in some cases not measurable. In this connexion it may be worth noting that the brass ring which carries the coil is fairly heavy, and that when it falls it produces an appreciable blow. Each of the three instruments has been subject to this shock hundreds of times, but has shown no sign of decay under it.

Perhaps I may be permitted to say that I do not advance the principle of the magnetic circuit as at all novel. It has already been embodied in several well-known applications. But I believe the idea is here made subservient to new purposes.

As a material embodiment of a standard of magnetism (magnetic lines, gaussses, or whatever else may be the right name) I find it helpful to many students. For this reason I propose to adjust future instruments of this size to a round number of lines, say 20,000 or 25,000, which will facilitate calculations arising out of their use.

The instrument is a most convenient standard for measuring magnetic quantities, whether it be the lines in any other magnet or the vertical and horizontal components of the earth's field. I am trying to use it for developing a new method of determining these quantities.

It enables me to simplify the magnetometer method for determining magnetic permeability. Over the ordinary magnetizing coil I wind a sufficient number of turns of a secondary.

The coil with its core is fixed in *any* position that is convenient, and the *relative* number of lines determined by the magnetometer deflexion in the usual way. At the end of the magnetization the total *absolute* number of lines (corresponding to the maximum magnetometer deflexion) is found by taking a throw from the secondary coil when the magnetizing current is reversed. This throw is then compared with that obtained from one of the inductors. By this method one is relieved from measuring distances whose cube &c. enters into the formula for reducing the observations.

I ought not to close without acknowledging the help given all through by the instrument-maker, Mr. G. Bowron, one of whose workmen, Mr. Collins, suggested the convenient method of release described in the paper.